Euclid's *Elements* and Philosophical Development

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The publication of Euclid's *Elements* about 300 B.C. represented a stunning accomplishment no less for Greek philosophy than for Greek mathematics. His text recapitulated two centuries of intense search for foundations of geometry carried on simultaneously and interactively with the quest for origins and method in philosophy. In company with the earlier Aristotelean logical and philosophical texts Euclid's synthesis represents the high point of Greek rationalism's drive for systematic perfection. And, like any philosophical classic, the *Elements* both set the lines of future development and posed in a subtle way questions which were to goad mathematicians and philosophers to this day. Along with E. S. Stamatis' recently completed second edition of Heiberg's critical text long out of print, an increasing number of contemporary studies calls our attention to an apparently inexorable line of conceptual development from Euclid to the present. Even a brief survey of these results can provide new insights into the interplay of philosophical and mathematical thought.

Let us first review the text of the *Elements*, considering its origins, and then see the way some of its questions have been developed in this century, paying attention to the interaction of mathematical and philosophical progress. The First Book of the *Elements* will be of special interest.[1]

Precious little is known of Euclid himself. Most of our knowledge comes from the late 3rd century A.D. mathematician Pappus and from the Neoplatonist Proclus (412-485).[2] The former reports that Euclid taught in Alexandria, the latter that he lived in the reign of

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1. *Euclidis Elementa,* vols. I-V 1-2 (Leipzig, 1969-1977), edited by E. S. Stamatis and based on the text (1883-1916) of J. L. Heiberg. Still immensely useful is Sir Thomas L. Heath's English translation and commentary *The Thirteen Books of Euclid's Elements.* (1956 Dover reprint of the 2nd (1926) edition published by the Cambridge University Press.)
2. Pappus, whose *Synagoge* or *Collection* is a guide to Greek geometry, mentions Euclid's students at Alexandria (vii, 35). Of his commentary on the *Elements* we have only an Arabic translation of the Commentary on Book X. (Arabic Text and Translation by W. Thompson. Cambridge, Mass. 1930. Harvard University Press). Proclus' *Commentary on the First Book of Euclid's Elements* is of great value for the history of Greek philosophy and mathematics. The Greek text has been edited by G. Friedlein (Leipzig, 1873, reprinted 1973, Teubner Verlag). There is an English translation by Glenn R. Morrow (Princeton University Press, 1970).

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Ptolemy the First (c.306-283), and that "by choice he was a Platonist and was versed in this philosophy." The *Elements* is not Euclid's only extant work. We have also the *Data, Sectio Canonis* (Elements of Music), *Division of Figures, Optics, Catoptrics,* and the *Phaenomena.* Knowledge of some lost works is also provided by Pappus and Proclus.

Our knowledge of the development of Greek geometry prior to Euclid depends significantly on a now lost history by Eudemus of Rhodes (fl. 320 B.C.), a student of Aristotle. Proclus adapted a summary of this history for his commentary on the First Book of the *Elements.* Eudemus recorded "(books of) Elements *(stoicheia)*" by Hippocrates of Chios (c. 410 B.C.), "the first of whom we have any record who did so." Hippocrates' text was followed by Elements composed by a certain Leon roughly contemporary with Plato. Later another book of Elements was published by Theudius of Magnesia, a member of the Academy in Plato's time. Euclid's text then appears as the end product of a sequence of works of which all have perished, eclipsed by the appearance of the Euclidean treatise. Proclus remarks that "Euclid . . . brought together the *Elements,* systematizing many of the theorems of Eudoxus, perfecting many of those of Theaetetus, and putting in irrefutable demonstrable form propositions that had been rather loosely established by his predecessors."[3]

Thirteen Books comprise the *Elements.* Two supplementary Books were included but scholars no longer attribute them to Euclid.

The First Book begins abruptly with "Definitions," "Postulates," and "Axioms" (or "Common Notions"), followed immediately by 48 propositions, deduced only, as Euclid apparently thought, from these starting points by logical means or rules of inference presumed known and accepted by his readers. Neither for this Book nor for the whole text is there stated purpose, historical survey, or suggestion of sources, prerequisites, or motivations. We shall discuss the Definitions and Postulates more in detail a bit later. The Propositions themselves fall into three groups. The first group (I 1-26) is mostly concerned with triangles, the second (I 17-32) includes the critical theory of parallel lines and the angle sum theorem for triangles, while the last group (I 33-48) studies parallelograms, triangles and squares in terms of areas. We shall see that there is a momentous division between propositions I 28 and I 29.

The Second Book continues the third section of the first. It constitutes a geometrical equivalent of our elementary classical algebra of

3. *Commentary,* ed. Friedlein, p.64-68. Eudoxus of Cnidos was a contemporary of Plato and, possibly, one of his students. Francois Lasserre has edited the fragments attributed to him (Texte und Kommentare 4, 1964).

quadratic equations. Its theorems are not all geometrical theorems but mostly algebraic ones in geometric form whose origins can be traced to Babylonian sources. The Third Book begins a study of circle geometry. Book Four introduces problems dealing with polygons inscribed or circumscribed to circles. A manuscript scholiast insists that this is all "the discovery of the Pythagoreans."[4]

Book Five details Eudoxus' theory of proportion. The Pythagorean cosmological teaching that all reality is whole numbers or ratios of whole numbers was crippled by the discovery that the ratio of the side of a square to its diagonal could not be expressed as the ratio of two whole numbers.[5] Eudoxus resolved the mathematical problem in a geometrical way, letting segments or areas represent magnitudes. He set on a firm foundation the geometry of proportion and similarity. The treatment of the problem in a purely arithmetical way had to await the 19th century response of Dedekind and Weierstrass to the philosophical challenge probing the foundations of differential and integral calculus. According to a scholiast (recorded in a 13th century manuscript) the arrangement and sequence of Book Five is that of Euclid himself. This general theory is applied to the geometry of similar figures in Book Six. The Book develops the Pythagorean "application of areas," a technique which is the geometrical equivalent of an algebraic solution of quadratic equations with at least one positive root. As in Book Two, the content is Babylonian in origin.

Books Seven to Nine are "arithmetical books." They study in a

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4. Heiberg-Stamatis, V 1, p. 204. Cf. also E. Neuenschwander, "Die ersten vier Bücher der Elemente Euklids," *Archive for the History of the Exact Sciences* 9 (1972/3) 325-380. His careful conclusions are that the contents of Books Two and Four and large sections of Book Three are to be attributed to the Pythagoreans and even that some results on Book One were available to them. O. Neugebauer suggested the links between Babylonian and Greek Mathematics in *Vorgriechische Mathematik* (Berlin, Springer Verlag, 1934. 2nd ed. 1969). Following Neugebauer's lead Bartel L. Van der Waerden studied the Pythagorean geometric transformation of Babylonian algebraic techniques in his *Science Awakening* (Groningen, Noordhoff, 1954. English translation by A. Dresden, with additions of the author. New York, Oxford University Press, 1961). Cf. Also O. Neugebauer, "Zur geometrischen algebra," *Quellen und Studien zur Geschichte der Mathematik, Astronomie, und Physik* 3 (1936) B 245-259.

5. Cf. W. R. Knorr, *The Evolution of the Euclidean Elements.* A study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry. (Boston, D. Reidel, 1975). He dates the discovery of the incommensurability of side and diameter of a square between 430 and 410 B.C. Cf. S. Unguru review in *Isis* 68, 2 (1977) 314-316. See also K. von Fritz, "The discovery of incommensurability by Hippasus of Metapontem," *Annals of Mathematics* (2) 46 (1945) 242-264.

geometrical way the properties of certain integers: primes, squares, cubes, perfect numbers, etc. Van der Waerden, developing a 1904 thesis of Paul Tannery, has established that the first 36 theorems of Book Seven were taken over by Euclid without substantial change from a Pythagorean text of the fifth century B.C. Also, he adduces ample reasons (in terms of its murky logical patterns) to attribute the main part of Book Eight to Archytas of Tarantum who introduced Plato to the exact sciences and the philosophy of Pythagoras. O. Becker observed that in Book Nine the last 16 theorems are nothing more than an appendix of Pythagorean teaching on even and odd numbers dating from the middle or first half of the fifth century B.C.[6]

Book Ten, an extensive unified theory of incommensurable line segments and often considered the most perfect book of the *Elements*, is linked with Theaetetus, the young Athenian killed in battle in 369 B.C., and honored by Plato's dialogue. In fact, parts of Plato's dialogue and the beginning of Book Ten are related to each other. For to prove the proposition he states at 147D - 148A Theaetetus needed the 5th, 6th, and 9th propositions of Euclid's Book. Both a scholiast and Pappus add that the last of these propositions was discovered by this mathematician, though his proof was surely different from Euclid's which depends on Eudoxus' theory of proportion not available to Theaetetus. This theory of irrational magnitudes, prompted by the Pythagorean impasse, developed by Theodorus of Cyrene and perfected by Eudoxus and Theaetetus may be considered one of the most significant achievements of Greek mathematics. Unhappily, despite references in Aristotle, we have little direct testimony of the various steps leading from Pythagoras to Euclid, and interpretation of both the Platonic and the Euclidean texts have vexed ancient and modern commentators alike. The most successful exegesis to date seems to be the recent work of Wilbur Knorr.[7]

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6. Cf. *Science Awakening*, p. 110-115, 152-155. Becker's paper "Die Lehre vom Geraden und Ungeraden im Neunten Buch der Euklidischen Elemente" appeared in *Quelle und Studien zur Geschichte der Mathematik* 3 (1936) B 533-553.

7. W. Knorr, *op. cit.* note 5, devotes the core of his study to a new interpretation of this passage. Cf. W. Thompson's *Commentary of Pappus on Book X*, cited in note 2, especially pp. 72-76, 180-184, and also Sir Thomas L. Heath, *A History of Greek Mathematics*, 2 vols. (Oxford, 1921), vol. I, pp. 204-208, 402-411. Van der Waerden, *Science Awakening*, pp. 165-172 attributes all of Book Ten to Theaetetus. The scholium mentioned is reproduced in *Heiberg-Stamatis*, V 2, p. 113, #62, and in the Loeb Classical Library Text *Selections Illustrating the History of Greek Mathematics*, vol. I, pp. 380-381 (I. Thomas, editor. London, 1939). The problem of

Books Eleven to Thirteen deal with three-dimensional or solid geometry. Book Thirteen aims to construct the five regular (Platonic) solids and to circumscribe each by a sphere. Proclus, with exaggerated enthusiasm for Plato, maintained that Euclid thought the purpose of the *Elements* as a whole was the construction of these figures mentioned by Plato in the *Timaeus*. Some evidence suggests that the Book was essentially written by the author of Book Ten, Theaetetus.

This extensive Greek activity Euclid surveyed and unified in his text. Unfortunately, since he recorded no trace of experimentation done to establish the procedures, definitions, postulates and axioms of his geometry, we have to look carefully elsewhere for such traces, and especially in the text of Aristotle.

Now to return to the First Book. Readers of the *Elements* often find difficulty in the very first definitions: a *Point* is that which has no parts, a *Line* is length without breadth. Not only is there seeming ambiguity in the formulations, but also the definitions do not and cannot function as definitions in the propositions and constructions justified by the postulates and axioms. For Euclid these definitions apparently served to clarify the nature of the idealized physical objects his geometry studied. And the assertions of mathematics were true and not just hypothetically correct. Mathematics, and geometry in particular, was of interest precisely because it asserted the truth of some aspect of extramental reality. It was only with David Hilbert and the shift to formalism in the late nineteenth century consequent on the discovery of noneuclidean geometry that the need for undefined and nonreferent basic terms was asserted and the hypothetical nature of mathematics proclaimed. Plato, in the *Republic,* and Aristotle in the *Posterior Analytics* had already established that any science needs unprovable first principles as starting points.[8] These are Euclid's Postulates and Axioms. How precisely they differ is disputed, but need not detain us here.

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determining what kind of evidence, if any, the story in *Theaetetus* 147d-148b provides for the actual historical developments in Greek mathematics is also studied by M. F. Burnyeat, "The Philosophical Sense of Theaetetus' Mathematics," *Isis* 69 (1978) 489-513. His purpose is to vindicate the essential historicity of the story more carefully than previously and to bring out the philosophical sense of the scene, that is, its contribution to the methodological preliminaries to Plato's inquiry into knowledge. His independent analysis generally agrees with Knorr's although it diverges on the important interpretation of 147d. A number of Knorr's interpretations of the history of earlier Greek mathematics depend crucially on this interpretation.

8. *Republic,* vi. 510b-c; *Posterior Analytics* I 10, 76a30-77a2. See A. Wedberg, *Plato's Philosophy of Mathematics* (Stockholm, Almqvist and Wikell, 1955). A useful collection and commentary on Aristotelean texts on mathematics is found in the posthumous work of Sir Thomas L. Heath, *Mathematics in*

The first four of Euclid's postulates are succinct: Be it postulated: I. To draw a straight line from any point to any point; II. To produce a finite straight line continuously in a straight line; III. To describe a circle with any center and radius; and IV. That all right angles are equal to one another.

The Fifth Postulate precipitated a two thousand year debate in mathematics and a crisis in philosophy. Euclid postulates: V. That if (A:) a straight line meet two straight lines so as to make the interior angles on the same side of it taken together less than two right angles, then (B:) these straight lines, being continually produced, shall at length meet on the side on which are the angles which are less than 2 right angles.

What is the Fifth Postulate problem? To begin, note that in terms of complexity alone the postulate stands suspiciously apart from the first four. Even more disturbing to the Greek geometers was that its converse was proved by Euclid in proposition I 17, as well as its contrapositive in I 29. Schematically the relation of the Fifth Postulate and these propositions is this:

P	V	: If A, then B	Proposition
I	29	: If not B, then not A	Contrapositive
I	17	: If B, then A	Converse.
I	28-29:	If not A, then not B	Contrapositive of Converse.

Proclus asks "Is it not ridiculous that theorems whose converses are demonstrable should be ranged among the indemonstrables?" and poses the problem when he insists that the Fifth Postulate both be struck from the list altogether and a proof be provided based on the first four postulates alone. Echoes of this dissatisfaction are also heard in his comments on the proof of I 29 for which for the first time Euclid uses (reluctantly?) the Fifth Postulate.

In many textbooks the troublesome postulate is replaced with an equivalent formulation: Through a point not on a given line, precisely one parallel line can be drawn. That is, Euclid's five postulates have this proposition as a logical consequence, and the first four postulates and this proposition have the fifth postulate as a conse-

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Aristotle (New York, Oxford University Press, 1949). For a discussion on how the Aristotelean axiomatics of *Posterior Analytics* I reflects certain aspects of the practice of Greek mathematicians as known to Aristotle from a study of a pre-euclidean work of Elements, see A. Gomez-Lobo, "Aristotle's Hypotheses and the Euclidean Postulates," *The Review of Metaphysics* 30 (1977) 430-439. This is also a criticism of the classical T. L. Heath and H. P. D. Lee thesis that the Aristotelean definitions match the definitions in Euclid, the axioms in Aristotle correspond to the Euclidean common notions and that the pendant to the Euclidean postulates is found in Aristotle's hypotheses.

quence. In this way, what is called the Fifth Postulate Problem is also called the Problem of Parallels. It is easily shown that the Fifth Postulate is equivalent to the proposition that the sum of the interior angles of a triangle is 2 right angles.

Proclus was not the first to pose the Fifth Postulate Problem or the Problem of Parallels. Indeed, he preserves for us (invalid) proofs by Claudius Ptolemy (c. 150 A.D.) of Postulate V and of I 29. He also mentions the efforts in this matter of his predecessor Geminus. After a criticism of Ptolemy's proof he proposes his own (invalid) proof of the disputed postulate, failing to observe that one of his own assumptions either itself needs Euclid's Postulate for proof, or being taken as a postulate, then becomes equivalent to the Fifth Postulate.

Although Aristotle, in the *Prior Analytics,*[9] had already pointed out the danger in this very task, this vicious circle situation will recur countless times in the research efforts of both Arabic and Western mathematicians until Gauss, Bolyai, and Lobachevskii simultaneously and independently resolve the Fifth Postulate Problem with the discovery of hyperbolic noneuclidean geometry in the early nineteenth century. Their geometry retains Euclid's first four postulates but replaces the parallel postulate with the contrary assertion that through a given point not on a specified line, at least two distinct parallel straight lines may be drawn.

Or better, perhaps, with the *rediscovery* of noneuclidean geometry. For we can see the perspicacity of Euclid in his recognition of the need for a postulate specifying the resolution of the parallels problem. Moreover, Euclid's unwillingness to use Postulate Five until I 29 means that his first twenty-eight propositions are also valid in hyperbolic noneuclidean geometry. The numerous results characteristic of Euclidean geometry and distinguishing it from hyperbolic geometry only begin with I 29. It is tempting here to say that Euclid was the first noneuclidean geometer! A bit more accurately, though, contemporary mathematicians say these are twenty-eight propositions of neutral or absolute geometry.

But let us now turn back the clock a bit more and search in the text of Aristotle where, as Professor Imre Toth has noted,[10] certain passages suggest the possibility of an early Greek "noneuclidean"

9. *Prior Analytics* II 16, 64b28-65a9. Cf. Heath, *Mathematics in Aristotle,* p. 27.

10. Imre Toth, "Das Parallelproblem im Corpus Aristotelicum," *Archive for the History of the Exact Sciences* 3(1966/7) 249-422. He provides a summary presentation in "Non-Euclidean Geometry before Euclid," *Scientific American* (1969) 221(5) 87-98. The original paper is reviewed by C. Nicasius, "La Conscience Malheureuse dans la Géométrie Grècque," *Archives de Philosophie* 31 (1969) 285-7.

approach to the problem of parallels. Toth observes that the critical proposition I 29 is incompletely proved by Euclid but that the necessary missing parts can be found in Aristotle's *Prior Analytics* II, 17 (66a 11-15), a text probably composed a good half century prior to the *Elements*. In form, Euclid's proof is indirect. Essentially, to get his conclusion, he must show that two angles A and B are equal. Using the *reductio ad absurdum* technique he employs Postulate V to show that if A be greater than B, then a contradiction ensues, and prematurely concludes that A and B are equal. Aristotle, however, provides the missing proof that if A be less than B, a contradiction also ensues, and thus the only possibility is that A and B are equal. These three cases correspond to the three possibilities for parallels, that is, through a given point not on a line there are many parallels, precisely one parallel, or none at all. If A is greater than B, then more than one parallel can be drawn, if A is less than B, no parallel can be drawn, while if A equals B, a unique parallel can be drawn. Contemporary mathematicians see each situation for parallels specifying (in modern terminology) a type of geometry: hyperbolic, elliptic, or parabolic (Euclidean). Aristotle's proof shows that on the strength of the first four Euclidean postulates elliptic geometry, the second case, may be ruled out. But it takes an explicit assumption of the Fifth Postulate, as Euclid indeed made, to rule out the first or hyperbolic case.

Thus Toth argues that the pre-euclidean geometers recognized that the case of elliptic geometry was inconsistent with even what would be the first four of Euclid's postulates, but what is more interesting, that there was seen a possibility of choice between the geometry of a single parallel and that of more than one parallel. Evidence is marshalled from texts of the Aristotelean school, the *Magna Moralia* and the *Eudemian Ethics*, which compare the influence of basic principles on ethical values to the influence of postulates on the essence of geometrical objects, for example, that in a triangle the sum of the interior angles is equal to 2 right angles, a property that is true of triangles only in Euclidean geometry.

In these texts the Euclidean situation is considered as hypothetical as the other two and all three situations are presented as being equally possible. The Aristotelean text admittedly hesitates at this possibility *(Eudemian Ethics II,* 6, 1222^b 23-41), no doubt because of the unexamined understanding that geometry was direct but abstract knowledge of our sensible physical world. So if, logically, there were several possibilities for a postulate specifying the nature of parallels, one chooses, freely, that postulate which is in accordance with physical reality at least as perceived in geometric diagrams. Toth remarks:

> As in (Aristotle's) ethics it is the ethical sense which decides what is right and what is wrong, so here in geometry the decision between principles is made by *nous*, the intellectual intuition, alone. It is entirely likely that this conception was a theoretical justification for the deadlock in which ended the attempts to demonstrate this fundamental proposition of Euclidean geometry (i.e. Postulate V) as a theorem of absolute geometry.

Toth is at pains to point out that Aristotle for all of his perplexity at the situation does not assert that propositions arising from geometries other than Euclidean are false. They are correct insofar as they do not derive from a logical error in the argumentation from first principles to conclusion. But the choice between two opposite geometries appears to Aristotle almost as an ethical one: choose the good geometry (the unique parallel case, Euclid's Fifth Postulate) in accordance with nature, or the wrong one (more than one parallel, the hyperbolic case) against nature.[11]

We do not know who explicitly first made the choice. Relying on Proclus' testimony, Toth conjectures that it was Euclid himself. At any rate, the prestige of Euclid's text confirmed this choice of a geometry which accorded with the observed physical reality of accurately drawn figures and whose need was perceived by geometers contemporary with Aristotle. The fact that there was a genuine mathematical choice was obscured until the nineteenth century. Because of the success and genius of his *Elements* Euclid provided a foundation for geometric development, especially by Apollonius and Archimedes. But not until the (re)discovery of hyperbolic noneuclidean geometry was there the possibility of understanding the ground and significance of the choice.

But let us now consider a more general source problem. For the genesis of the systematic deductive method found in the *Elements* is still shrouded in mystery. It most certainly appears, like philosophy itself, to be an original Greek accomplishment. It too seems to be linked with philosophical method. Widely varying hypotheses have been offered. The Soviet mathematician A. N. Kolmogorov argued that the change in the character of mathematics from the practical computational rules of Egyptian and Babylonian technology can be attributed to the advanced socio-political and cultural development of the Greek city-states in the fifth century B.C. This generated a high level of the art of dialectics and the Sophist phenomenon. In addition, the genesis of philosophical thinking independent of religion showed the need to account rationally for natural phenomena especially in the face of difficulties created by the Sophists. In turn

11. Toth, *op. cit.*, p. 98.

these developments gave mathematics new goals and methods. The Euclidean axiomatic method was the result.[12]

Van der Waerden specifically put the beginning of the deductive method with Thales (624-547), who probably had the task of discriminating between variant geometric volume and area formulas from Egyptian and Babylonian traditions. Thales determined the correct formulas and fitted them into a logically connected system. Van der Waerden asserts:

> This is exactly what he did, according to Eudemus, and it is exactly as the beginning of such a logical system that one may expect to find such Irish bulls as: vertical angles are equal, the base angles of an isosceles triangle are equal, a diameter divides a circle into two equal parts, etc.[13]

Sir Thomas L. Heath, in his monumental *History of Greek Mathematics,* also held this view. However, it is not widely accepted today because of skepticism about the accuracy of the Eudemian testimony on which it depends.

More conservatively, Otto Neugebauer and Kurt von Fritz place the origin of deductive form much later and consequent on the fifth century Pythagorean discovery of the incommensurability of side and diagonal of a square. For them the development of deductive mathematics and its formulation on definitions and postulates are linked with the birth of Aristotelean logic. Morris Kline explicitly attributes such formulation to Eudoxus as a result of his research in the theory of incommensurable ratios preserved, as noted, in the Fifth Book of the *Elements.*[14]

Francois Lasserre uses Plato's testimony (147 D) that Theaetetus was the first to understand that a mathematical theory develops from definitions which are broad enough to contain within them the solutions of all the problems posed in such a theory. He concludes that:

> It is not before Leon, the second author of Elements (in point of

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12. *Great Soviet Encyclopedia,* s.v. Mathematics. (The 3d edition (1970, English translation 1973) article "Axiomatic Method" by I. A. Gastev and E. S. Esenin-Volpin does not treat this question in detail.)
13. *Science Awakening,* p. 89.
14. Cf. Otto Neugebauer, *The Exact Sciences in Antiquity* (2nd edition. Providence, Rhode Island, Brown University Press, 1957. Paperback reprint, Harper Torchbooks, New York. 1962), pp. 147-9. In addition to his paper cited in note 5, cf. also K. vonFritz: "Die APXAI in der griechischen Mathematik," *Archiv für Begriffsgeschichte* 1 (1955) 13-103. (Reprinted in his *Grundprobleme der Geschichte der Antiken Wissenschaft,* Berlin, De Gruyter. 1971). Morris Kline: *Mathematical Thought from Ancient to Modern Times* (New York, Oxford University Press, 1972), p. 50.

> time), that the ideas of an axiom, a postulate, an hypothesis — in short, the first principles of mathematics — are acknowledged and defined . . . The initial attempt at a generalization which embraces . . . specific sciences in its turn and which can include all mathematics under the same laws, dates only from Eudoxus of Cnidus.[15]

The Hungarian scholar Arpad Szabo has developed an extensively detailed but controversial hypothesis about this transformation of mathematics into a deductive science based on definitions and axioms.[16] His argument looks to the transition of Greek mathematics from Pythagorean arithmetic and number theory to geometry in the context of the dialectic of the Eleatic philosophers Parmenides and Zeno. Szabo's textual study purports to show that arithmetic itself was an independent further development of the Eleatic philosophy of being, with number an intellectual multiplication of the unity of being. This philosophy provided adequate range for arithmetic once the essentially new definition of number had been introduced by the Pythagoreans. The definition itself was not a basis for opposition between Pythagoreans and Eleatics. Indeed, Szabo argues

> (the definition) proved so productive that it allowed further extension of the Eleatic method and the construction of a non-contradictory discipline — arithmetic — which almost seemed to be a new and independent province of Eleatic philosophy. The Pythagorean arithmetic has become the greatest and most lasting creation of Eleatic philosophy.[17]

But this success could not be duplicated in geometry. A paradox of Zeno, as recorded by Aristotle,[18] forced the geometers along another line of development, Zeno asserted "Half the time equals its double." He attempted to prove this by his characteristic and pioneering *reductio ad absurdum* technique which was adopted by mathematicians and later appeared with great effectiveness in the

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15. F. Lasserre, *The Birth of Mathematics in the Age of Plato* (Larchmont, N.Y. American Research Council, 1964. H. Mortimer, tr.)
16. "Anfänge des Euklidischen Axiomensystems," *Archive for the History of the Exact Sciences* 1 (1960) 37-106, "The Transformation of Mathematics into Deductive Science and the Beginnings of its Foundation on Definitions and Axioms," *Scripta Mathematica* 27 (1964) 27-49, 113-139; "Greek Dialectic and Euclid's Axiomatics," in I. Lakatos (ed.), *Problems in the Philosophy of Mathematics* (Amsterdam, North Holland, 1972), pp. 1-27.
17. In *Scripta Mathematica* (note 16), p. 137.
18. Aristotle, *Physics* VI 9 239b-240a18. Cf. Heath, *Mathematics in Aristotle*, pp. 137-140.

Elements. The paradox is of interest especially since the concepts "half" and "double" can be easily replaced by the concepts "part" and "whole." Szabo argues that Euclid's Eighth Common Notion or Axiom "The whole is greater than the part" was formulated as an axiom precisely because someone using paradoxical argumentation like Zeno's contested its truth. When the consequences of this position were seen, the geometers were compelled to accept the original statement as a theorem evident in itself, but not capable of being proved. It became an axiom for them.

Thus the line of demarcation was the axiom, an empirical truth that could not be proved, could apparently even be refuted in some non-geometrical cases, but had to be chosen as the basis of further demonstration. The foundations of geometry, for all of being influenced by Eleatic philosophy, took a view antithetical to this philosophy.

Szabo's proposals have been challenged by a number of perceptive critics. University of Chicago professor Ian Mueller[19] views Szabo's hypothesis that Parmenides had a central position in the history of mathematics and that the change from 'empirical' to 'pure' mathematics is closely connected with the idealistic, antiempirical character of Eleatic and Platonic philosophy as unnecessary. The conception of Greek mathematics in its developed state on which these hypotheses rest is not justified by the character of the *Elements* as Mueller analyzes it. Thus he argues that the derivation of a Euclidean proposition is an experiment performed on idealized physical objects. The experiment is limited by preliminary agreements (first principles: definitions and axioms) about the nature of the objects, some of their properties, and the operations that can be performed on them. Admittedly, Euclid's argumentative procedure was not empirical in the sense that Babylonian and Egyptian procedure was, but it can be explained without reference to antiempirical philosophical movements. Moreover, Euclid's reasoning is logically correct, but even so, Mueller adduces evidence that the geometer did not know (or at least use) the formal logic of his time. Thus transformations of Euclid's arguments into modern logic are misleading. Measured by standards of contemporary mathematical logic the intuitive Euclidean arguments are not conclusive. But to infer that they are wrong is to make the unjustified generalization that intuition is always inconclusive in mathematics. Referring to Wittgenstein's initial statement in *Remarks on the Foundations of Mathematics,* Mueller reminds us that 'conclusiveness' can be de-

19. Ian Mueller, "Euclid's Elements and the Axiomatic Method," *British Journal for the Philosophy of Science* 20 (1969) 289-309.

fined in terms of mathematical practices as well as in terms of a mathematical ideal.

Other critics also raise doubts about the Eleatic origin of the Euclidean axiomatic method. Possibly Greek mathematicians recognized a valid pattern of reasoning before the philosophers certified it as such. But this objection assumes a separation of mathematics and the philosophy of nature not present in the Pythagoreans and Eleatics who were taken up with the mathematical solution of cosmological problems.[20]

A clearcut resolution probably cannot be had, but Szabo has provided insightful hypotheses to explain events of which we have no direct witnesses.

Let us move on to see the development and influence of the Euclidean problem of parallels which was, as remarked, definitively settled by nineteenth century geometers, following on Arabic and Latin attempts to resolve the problem.

An interesting part of this history involves a trio of Jesuits: Christopher Calvius (1539-1612), Gerolamo Saccheri (1667-1733), and Roger Boscovich (1711-1787). Clavius, professor of mathematics at the Roman College of the Society of Jesus, published a Latin version of the *Elements* whose first edition appeared in 1574. It was not a translation, but rather a reworking of Euclid's proofs. Along with a treasury of notes gotten from earlier editors and commentators there are (metamathematical) studies of Euclid's forms of proof. Clavius approached the parallel problem understanding parallels as equidistant straight lines, though he recognized the need to justify that the locus of a set of points equidistant from a given straight line is itself a straight line. His attempt at proof failed, and we know now that such a proof either demands Euclid's Fifth Postulate or some equivalent of it.[21]

Clavius was also the author of influential text books in algebra and geometry. René Descartes reflects his use of them as a student at the Jesuit College of LaFléche. These texts were studied by Saccheri too, whose attempt to prove the Fifth Postulate in his treatise *Euclid Freed of Every Stain* was, unintentionally, the modern prelude to noneuclidean geometry.[22]

20. For comments of W. C. Kneale, L. Kalmar, A. Robinson, et al., see *Problems in the Philosophy of Mathematics,* p. 9-20, cited in note 16.
21. There is no full length study of Clavius as mathematician, but Heath, in *The Thirteen Books of Euclid's Elements,* provides evaluation and excerpts from Clavius's presentation of Euclid. See also Joseph MacDonnell, "Jesuit Mathematicians Before the Suppression," *Archivum Historicum Societatis Iesu* XLV (1976) 139-147.
22. A text and translation have been provided by George Bruce Halsted, *Girolamo Saccheri's Euclides Vindicatus* (Chicago, Open Court Press. 1920).

A mathematics professor at the Jesuit College of Pavia, Saccheri was also a student of logic, philosophy, and theology. His profound and pioneering 1697 study *Logica Demonstrativa* treats complex definitions (ones which presuppose the existence of an entity satisfying conditions whose compatibility has not been proved). These observations seem motivated by attempts such as Clavius made to replace the Fifth Postulate by understanding parallels as equidistant straight lines. Saccheri insists that before such a definition can be used, one must prove that the geometric locus of points equidistant from a straight line is itself a straight line. Anticipating the distinction made in 1843 by J. S. Mill in his *System of Logic* between real and nominal definitions, Saccheri develops the notions of *definitio realis* and *definitio nominalis*. The latter explains the meaning to be given to a term, the former both declares the meaning of a term and affirms the existence of the thing defined (or, in geometry, the possibility of its construction). Such definitions are ordinarily obtained as the result of a sequence of demonstrations. Saccheri illustrates this with Euclid's construction of a square in *Elements* I, 46. For although Euclid defines a square at the beginning of Book One, he never presumes in his argumentations the existence of a square as defined until he has constructed one without the assumption of its existence.[23]

Saccheri tried to use a *reductio ad absurdum* proof to establish the parallel postulate. Although he necessarily failed to carry his argument to the conclusion he wanted, still he derived a sequence of theorems now recognized to be the initial propositions of a consistent noneuclidean geometry. Saccheri seems to have been so intent on establishing that Euclid's was the only logically acceptable geometry that he failed to recognize what he had actually come upon. He tried to derive a contradiction where none could be found and lost the opportunity to become the modern discoverer of noneuclidean geometry. His research was published in the year of his death 1733, and very largely dropped from sight. Only by chance was the book rediscovered in the nineteenth century by the geometer Eugenio Beltrami who recognized its significance.[23a]

The Dalmatian Roger Boscovich was a later successor of Clavius to

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Saccheri's works are listed in Sommervogel, *Bibliotheque de la Compagnie de Jesus*, 1896, vol. 7, p. 360. Additional bibliographical material is found in the doctoral dissertation of Arnold F. Emch in the Harvard College Library.

23. Emch published an extensive analysis of the text in "The *Logica Demonstrativa* of Girolamo Saccheri," *Scripta Mathematica* 3 (1932) 51-60, 143-152, 221-233.

23a. Cf. L. Allegri, "Book II of Girolamo Saccheri's Euclides ab omni naevo vindicatus," *Actes du dixième congrès international d'Histoire des Sciences* the chair of mathematics in the Roman College, and to Saccheri at the College of Pavia.[24] But the extent to which he was familiar with their work on the Euclidean problem is not clear. Even if he treated foundational problems of geometry only in passing, still there are suggestions in his papers that unlike Clavius or Saccheri he positively discerned the possibility of a noneuclidean geometry, and J. F. Scott cites passages indicating that Boscovich had an attitude not far removed from the later noneuclidean geometers.[25] The Boscovich archives at Berkeley may well provide documents clarifying his views.[26]

The many efforts to resolve the problem of parallels came upon success through the simultaneous but independent work of Gauss (1777-1855), Lobachevskii (1793-1856) and Janos Bolyai (1802-1860).[27] Gauss, however, did not publish his conclusions. Lobachevskii announced his results in 1826 at Kazan University and published them in the *Kazan Messenger* for 1829. He dared to present a geometry with a postulate contradicting the parallel postulate but exhibiting no inherent logical contradiction, and explored the consequences in three treatises produced from 1835 to 1855. Janos Bolyai, following the research of his father, reached the same conclusions in 1829 as Lobachevskii had, publishing them in 1832 as an appendix to his father's treatise on the topic. But after an 1840 German translation of Lobachevskii's results, the temperamental Janos Bolyai stopped his own publications, leaving the field to Lobachevskii and his followers.

Subsequent geometers slowly became aware of these developments along with the problems posed for mathematics and for philosophy. How were Euclidean and noneuclidean geometries related, if at all? A startling conclusion soon emerged. An ingenious construction showed that if the postulates of Lobachevskii's noneuclidean geometry led logically to contradictory theorems, then so did the postulates of Euclidean geometry.

(1962) vol. 2, p. 663-665 (published 1964). Ms. Allegri lists citations of Saccheri's discussion of Euclid's treatment of proportions to give a key for estimating the dispersal of Saccheri's book. See also Emch, *loc. cit.*, p. 52.

24. A biography and technical surveys are given in *Roger Joseph Boscovich, S.J., F.R.S., 1711-1787*. Studies of his life and work on the 250th anniversary of his birth. Edited by Lancelot Law Whyte. (London: Allen and Unwin, 1961).

25. *ibid*. pp. 187-190.

26. A description of the sizable collection of Boscovich papers purchased by the University of California is given by Roger Hahn, "The Boscovich Archives at Berkeley," *Isis* 56 (1965) 70-78. The Library of the American Philosophical Society in Philadelphia has microfilm copies of this collection.

27. A still useful reference is R. Bonola, *Non Euclidean Geometry* (New York, Dover reprint of 1912 edition. H. S. Carslaw, tr).

Had Saccheri succeeded in proving the inconsistency of hyperbolic geometry he would have destroyed Euclidean geometry. In short, Euclidean and hyperbolic noneuclidean geometry stand and fall together. Even more, because Euclidean geometry can be reduced to algebraic formulations by the techniques of Descartes' analytical geometry, the consistency of Euclidean geometry became equivalent to that of algebra, and ultimately to that of arithmetic. Was it then possible to prove that the postulates and theorems of arithmetic as formalized by Peano (1858-1932), for example, were contradiction-free? Significant efforts were made to resolve the problem, as well as to resolve the philosophical problem of determining the relationship of mathematics to physical reality. Geometry began as a study of physical reality; now antithetic geometries were available and would soon be applied to describe the one physical reality. But if the question of Euclid and/or Lobachevskii challenged the mathematicians, the challenge was not limited to them.

For the nineteenth century development of noneuclidean geometry precipitated a philosophical crisis especially for Kantians. One reason why Kant developed critical philosophy was his interest in providing a secure philosophical foundation for the universal and necessary character of the Newtonian laws of nature. Now Newton's physics was expressed and developed in terms of Euclidean geometry. Indeed, in the era of Newton and Kant pure and applied mathematics were not distinguished, nor were mathematics and physics separated.

A common understanding is that Kant tried to explain and ground the transcendental character of the laws of physics by requiring that all experiences be perceived under the *a priori* space form of sensibility. This space form is a Euclidean mold. So if our conception of space, as it is developed in geometry and physics, is had from our sensible intuition of space, it follows that our conception of space is necessarily that of Euclidean space. How then does the Kantian philosopher come to grips with mathematically acceptable geometries which are not Euclidean? Do these undermine the Kantian structure of *a priori* forms of sensibility?

Many mathematicians and philosophers have thought so. The distinguished Dutch mathematician and philosopher of mathematics L. E. J. Brouwer (1882-1966) admitted that the most serious blow to the Kantian system was the development of Lobachevskian and Riemannian geometries.[28] The logical positivist Hans

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28. L. E. J. Brouwer, "Intuition and Formalism," *Bulletin of the American Mathematical Society* 20 (1913/4) 85.

Reichenbach (1891-1953) argued that the different geometries that physicists after Einstein employ in their discussion of the universe cannot be satisfactorily reduced to Euclidean geometry. Hence Kant's assertion of a Euclidean world of phenomena is untenable.[29] Similar argumentation is developed by another Vienna Circle mathematician and critic, Hans Hahn. Experiment, not a *priori* necessity, tells which geometry (or geometries, in a physically nonhomogeneous universe) can best describe the physical world.

Contemporary Kantian scholars nevertheless insist that careful reading of Kant's 1787 *Critique of Pure Reason* will show noneuclidean geometry not necessarily precluded. Professor L. W. Miller of New Orleans has argued that quite generally in appraising Kant's philosophy of mathematics we must include the Analytic-Discipline account which supplements and corrects the Aesthetic account.[30] In particular, though the Aesthetic account of geometry now seems untenable, the Analytic account of mathematics as the science of construction of concepts in pure intuition is not undermined by the discovery of noneuclidean geometries insofar as these constructions may be Euclidean or noneuclidean and still be applicable to experience. Norman Kemp Smith had criticized Kant's theory of space as defective judged in the light of later geometrical developments because of his isolation of the *a priori* of sensibility from the *a priori* of understanding. Miller claims that for Kant this isolation was only provisional:

> If determined space is independent of the understanding then it limits the understanding by its determination and alternative geometries seem impossible, but if as in the Analytic account this determination is due to the understanding then alternative geometries are possible.[31]

Relying on Kant's remarks in #38 of the *Prolegomena to Any Future Metaphysics* that space determination is due to the understanding, Miller concludes that in the Analytic account Kant does allow for the possibility of both Euclidean and noneuclidean geometries. His paper ends with a thesis that contemporary mathematical research (and the theory based on it) supports rather than overturns the full Kantian doctrine. Mathematics as practised (discovery) and mathematics as theoreti-

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29. Hans Reichenbach, *The Rise of Scientific Philosophy* (University of California, Los Angeles. 1951). Chapter 8, "The nature of geometry").
30. Larry W. Miller, "Kant's Philosophy of Mathematics," *Kant Studien* 66 (1974/5) 297-308.
31. *ibid.*, p. 301.

cal deductive scheme (demonstration of proof) apparently fit the Kantian distinction between mathematical judgments and mathematical propositions.

Miller's analysis complements that of Professor Stephen Körner who had addressed two related difficulties.[32]

First, after observing that for Kant pure mathematics is synthetic *a priori* since it is about space and time, Körner remarks that Kant will not allow that a full description of the structure of space and time requires mere passive contemplation. Rather it presupposes the activity of construction. "To construct a concept" in Kantian philosophy is to go beyond presupposing or recording its definition. It is to provide it with an *a priori* object. In Kant's words: "To construct a concept means to exhibit *a priori* the perception which corresponds to the concept." But this does not mean to postulate objects for it as a contemporary mathematician might postulate the existence of a fifteen dimensional cube. We can, however, construct a three dimensional cube. The construction is possible not only because "three dimension cube" is a consistent concept but also because perceptual space is what it is. The construction is not a physical one, of course, but a physical construction is based on the possibility of an *a priori* one.

Körner calls our attention to Kant's distinction in the Introduction to the second edition (1787) of the *Critique of Pure Reason* between the thought of a mathematical concept requiring only internal consistency and its construction which demands that perceptual space have a certain structure. Thus Kant does not necessarily deny the possibility of self-consistent noneuclidean geometries.

There is however a more difficult second problem posed by the use of four dimensional Euclidean or noneuclidean geometry in special and general relativity theories. Körner allows that Kant was mistaken in assuming that perceptual space is described by three dimensional Euclidean geometry, but proceeds to argue that it is described by neither Euclidean nor by noneuclidean geometry.

Körner in fact argues that pure mathematics is disconnected from perception and that in mathematicizing perceptual concepts, statements, and theories, we so modify the perceptual concepts that they cease to be perceptual. The modification or idealization is essentially a 'disconnection' from perception. As for applied mathematics, Körner holds that the 'application' to perception of pure mathematics consists in a regulated activity involving the

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32. S. Körner, *The Philosophy of Mathematics* (Hutchison University Library, London. 1960), p. 28 ff.

replacement of empirical concepts and propositions by mathematical ones, the deduction of consequences from the mathematical premises thus provided, and the replacement of some of the deduced mathematical propositions by empirical ones.[33]

Körner's essay is representative of the regenerated interest in the philosophical problem of the nature of mathematics and of its relation to physical reality following the realization that neither the Kantian scheme nor the older Aristotelean explanation in terms of abstraction seemed capable of dealing with the proliferating geometries and their new found applications to the study of the physical world. Two major lines of development should be noted here.

England was the scene of sustained efforts by Alfred North Whitehead and Bertrand Russell to reduce all mathematics through the theory of sets or classes to an axiomatized symbolic logic. In this they continued the work of George Boole, C. S. Peirce, and Gottlob Frege, and their efforts culminated in the *Principia Mathematica* of 1910. (Due to technical difficulties, the section dealing with geometry and mainly the work of Whitehead, never appeared.) Russell in particular was not only quite confident that this approach could set mathematics on an unassailable foundation but also remarked that:

> the proof that all pure mathematics, including geometry, is nothing but formal logic, is a fatal blow to the Kantian philosophy . . . The whole doctrine of *a priori* intuitions, by which Kant explained the possibility of pure mathematics, is wholly inapplicable to mathematics in its present form.[34]

The two mathematicians presented a collection of primitive axioms expressed symbolically which they presumed a logician would accept as correct and on which they hoped to base the whole of mathematics. From these axioms, using only explicitly stated rules of inference, they move to derive other laws of logic such as the law of the excluded middle, the law of the double negative, and the method of proof by *reductio ad absurdum.* From logic comes the theory of classes and relations which in turn provides the foundation for a definition of cardinal (counting) number and the rules of ordinary arithmetic. A uniform and systematic development supplied the basic theorems of the arithmetic and analysis of real numbers. Such was the program of the Logicist School.

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33. *ibid.,* p. 182.
34. "Mathematics and the Metaphysicians" (written in 1901 and reprinted in *Mysticism and Logic,* Doubleday Anchor reprint of 1917 edition) p. 91.

On the continent, David Hilbert, starting from fundamentally Kantian presuppositions, proposed a formalist theory.[35] Mathematics, or a mathematical structure, essentially consisted in a collection of undefined symbols, of rules of combination of the symbols into formulas, of axioms or primitive formulas linking the undefined symbols, and of a set of transformation rules which when applied to the axioms generated new formulas or theorems. Though theorems derive from axions by logical transformation, neither axioms nor theorems are principles of logic. Models, interpretations, or realization of the structure in physical terms might be supplied, but are not part of the mathematical structure. Hilbert's ideas are illustrated by the axiom system he had already developed in 1899 to replace Euclid's system whose structural deficiencies became painfully obvious in the aftermath of Bolyai and Lobachevskii.[36] "Point," "line," "plane," "on," "between," . . . all these are undefined terms linked by such axioms as "There are at least three points not on the same line." Euclidean theorems follow from the axiom set by explicit transformational rules of logic. We can of course visualize or interpret these geometric statements in the usual way learned in school, but it is important to realize that there are also other models or visualizations of these statements. David Gans has described a model of Euclidean geometry in which the term "straight line" appears visually as a semiellipse in the interior of a circle.[37] All the familiar Euclidean statements about triangles, rectangles, and circles, can be consistently visualized in this way. Initially awkward to use, Gans' model or interpretation of the Euclidean plane has subtle advantages for geometric study. Other models have also been developed.

Although Hilbert allowed into mathematics Georg Cantor's recent (1879-1897) and controversial theory of sets, suitably axiomatized and admitting transfinite constructions as Kantian ideal elements, he proposed to control such a formalized structure

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35. See H. B. Curry, *Outlines of Formalist Philosophy of Mathematics* (Amsterdam, North Holland, 1958). Also G. Kreisel, "What have we learnt from Hilbert's second problem?" in *Proceedings of Symposia in Pure Mathematics*, vol. 28, pt. 1 (American Mathematical Society, Providence, Rhode Island, 1976).

36. David Hilbert, *Grundlagen der Geometrie* (Leipzig, Teubner. 1899) English translation: *Foundations of Geometry*, 2nd edition translated from the tenth German edition (LaSalle, I11. Open Court Press. 1971).

37. David Gans, "A circular model of the Euclidean plane," *American Mathematical Monthly* 51 (1954) 26-30. See also his "Models of projective and euclidean spaces," *ibid.* 65 (1958) 749-755.

by a metamathematical apparatus operating outside the structure and employing only finite procedures. In this way he hoped to prove the consistency or freedom from consequential contradiction of the formal system and, in particular, the consistency of the arithmetic of real numbers which grounds classical analysis and mathematical physics. He had also to show complete formalization. That is, every statement provable within the formal structure codifies a true proposition relative to the intended interpretation (for example, of the real numbers), and every true proposition is codified in a provable statement. Such was the program he set for the Formalist School.

What success came to the ambitious projects of the two schools? Technical difficulties and paradoxes in class theory forced Whitehead and Russell to use a "reducibility" axiom. Few logicians have found its form and content as a primitive axiom of logic completely acceptable, though subsequent study, especially by F. P. Ramsey, indicates that many of the objections can be muted. But an even deeper objection was raised. What right did Whitehead and Russell have to think that their project was valid? Granted that they had taken previously scattered and heterogeneous mathematical material and correlated it within the ambit of their logical system, what reason did they have for believing that all possible theorems of mathematics would likewise fall within this ambit? Did they have the same unexamined assumption as Euclid who seemingly thought that all true geometric statements could be reduced to his axiomatic starting points? In 1919 Emil Post developed the idea of the completeness of a formal mathematical system. Such a system is *complete* if it is not possible to add an axiom independently of those already prescribed, that is, an axiom not logically derivable from them, the undefined terms remaining fixed. Post was also able to establish that in a certain subsection of the *Principia Mathematica* any proposition seen as true under a specified interpretation or model of its symbols could be proved using the axioms and rules of inference which the system explicitly provided. Post's splendid work raised hopes that all of the *Principia Mathematica* could be treated with such techniques.

Unhappily, the economic depression of the Thirties was matched with the mathematical depression of the Logicists and Formalists. A young Austrian, Kurt Gödel (1906-1977), developing Post's ideas, proved two shattering theorems which still form the cornerstone of twentieth century logic and metamathematics. In October 1930 he presented to the Vienna Academy of Sciences an abstract "Some metamathematics results on completeness and consistency" of his paper "On formally undecidable propositions of *Principia Mathematica* and related systems I," to be published in 1931.[38] In a note (added in 1963) Gödel discusses his results:

> In consequence of later advances, in particular of the fact that due to A. M. Turing's work a precise and unquestionably adequate definition of the general notion of formal system can now be given, a completely general version of Theorems VI and XI is now possible. That is, it can be proved rigorously that in every consistent formal system that contains a certain amount of finitary number theory there exist undecidable arithmetic propositions and that, moreover, the consistency of any such system cannot be proved in the system.

In short, by Gödel's Theorem VI the *Principia Mathematica* is necessarily incomplete. There will be legitimately formed arithmetical propositions of the system such that neither the proposition nor the denial of the proposition can be proved within the system. And similarly for any axiomatic formulation of Euclidean geometry. Could such an unprovable proposition then be added to the system as an axiom inadvertently omitted by its formulators? Surely, but then, as Gödel's result shows, the augmented system would in turn allow another undecidable proposition. Even if an infinite sequence of undecidable propositions was added at one move to the original system, the resulting system would still be formally incomplete.

Hilbert's program of devising a consistency proof for a formal arithmetic system as seen in *Principia Mathematica* also comes to grief. Now it is relatively easy to show that if the *Principia Mathematica* (or any formal system, for that matter) is inconsistent, then *any* statement in the system can be proved. Inconsistency implies completeness. But Gödel's Theorem XI demonstrates that if a system as complex as the *Principia Mathematica* be actually consistent, then it is not possible to produce a valid proof of this within the system itself. He is however careful to observe that a finitary consistency proof may be possible which cannot be expressed in the formalism of the system. Should we formalize such a logical apparatus to a metasystem, we are again presented with the consistency problem for the metasystem, and the specter of an infinite systematic regress hovers over our efforts. If we employ the logic of ordinary discourse to establish consistency, we

38. K. Gödel, "Über formal unentscheidbare Sätze der *Principia Mathematica* und verwandterSystem I," *Monatshefte für Mathematik und Physik* 38 (1931) 173-198. English translation in J. van Heijenoort, *Frege and Gödel. Two Fundamental Texts in Mathematical Logic* (Cambridge, Harvard University Press, 1970).

note with A. Church that such an intuitive argument must involve a principle incapable of being formalized within the given system.[39]

Like Heisenberg's Uncertainty Principle in physics which puts a limit on the amount of information to be obtained from a physical system, Gödel's result suggests a limitation on the amount of information to be obtained from a formal axiomatic system. His conclusions may also be taken to suggest that the logic or ordinary discourse needed to ground the consistency of a formalized logic must be more than a game played with a set of symbols and inference rules, and must somehow itself be grounded in reality. Almost innumerable implications and questions remain. Does the problem of limitation arise from the formalism method itself, or, if we are mathematical Platonists, from the transcendent nature of the mathematical objects themselves? Or are there any mathematical objects which we do not ourselves explicitly and repeatedly construct? And if we construct mathematical objects ourselves, how can mathematics be uniform and consistent . . .

It is not here possible to list and explore the developments in mathematics, logic, and epistemology consequent to the Gödel theorems and the pervasive way they have changed our views of these disciplines. But it should be possible to understand the Incompleteness and Consistency theorems as the inevitable result of the publication of the *Elements*. Mathematics and philosophy were twin products of the miracle of Greek rationalism. The history of their adolescence and maturity sees them still as closely linked members of a family. Euclid wrote no less for philosophers than for mathematicians. His *Elements* will remain a landmark text in the history of western philosophy.

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39. Cited in L. O. Kattsoff, *A Philosophy of Mathematics* (Iowa State University Press, Ames, 1948), p. 194. See also W. Kuyk, *Complementarity in Mathematics* (Boston, D. Reidel. 1977) p. 49-55.