Mathematics as Philosophy: Barrow and Proclus†

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A defining feature of the early modern transformation of the sciences was a question: what role should mathematics assume, both as the language of natural philosophy and the source—if at all—of its principles? The relative centrality of that role was in dispute, as was the dialect of the language and the character of the principles. But the question itself helped to define a whole range of early modern issues of philosophical debate, from the ethic of enquiry to proper philosophical method, and from ontology to the right order of teaching and learning. Certainly for the first half of the period, Proclus' *Commentary on the First Book of Euclid's Elements* was an authority within that debate.[1] As the early modern period progressed, however, the relevance of ancient philosophical sources—including Proclus—to the methods and content of mathematical and natural enquiry came under increasing question. Historians generally see Francis Bacon as representative of the beginning of this questioning, at least as it was formulated in England. Indeed, Bacon included Proclus amidst the old philosophical authorities whom one needed to get over in order to study nature in a "pure" way:

> We have as yet not natural philosophy that is pure; all is tainted and corrupted; in Aristotle's school by dialectic; in Plato's by theology; in the second school of Platonists such as Proclus and others, by mathematics, which ought only to give definiteness to natural philosophy, not to generate or give it birth.[2]

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1. It was with reference to this wide range of influence that Alistair Crombie refers to the *Commentary* as "the most complete ancient account of the mathematical sciences," and one which was "to become the most influential early modern *scenario*" (emphasis mine); *Styles of Scientific Thinking in the European Tradition*, vol. 1 (London, 1994) 285.

2. *The Works of Francis Bacon*, ed. J. Spedding, R.L. Ellis, and D.D. Heath, vol. 4 (Boston, 1860–1864) 93.

The completion of that negative turn is usually represented in the figure of Sir Isaac Newton, for example in his famous dismissal of the language of the schoolmen and their "substantial forms" and "occult qualities." In doing so, Newton declared the mathematical principles of natural philosophy, and apparently *defined* for posterity "the whole business of philosophy" as a process of inferring the forces of nature mathematically defined from the phenomena of matter in motion.[3] Historians of science and philosophy have written much concerning the fortunes and transformations of ancient sources generally, from the beginning to the end of this period. Less has been said concerning the fortunes of Proclus' *Commentary*.[4] This is true even of Ernst Cassirer—surprisingly so, given the interest shown in Proclus' system by contemporary neo-Kantians such as Nicholas Hartmann.[5]

But at least for Isaac Barrow (1630–1677), an avid reader of Bacon, and immediate predecessor to Newton as the first Lucasian Professor of Mathematics at Cambridge, the *Commentary* played an important—albeit ambiguous—role.[6] On the one hand, Barrow's first three years of Lucasian lectures illustrated a feature of the later period's attitude towards ancient sources. As an introduction to the nature and methods of mathematics, these lectures asserted that mathematics itself provides the starting points and the solutions to questions which had earlier been handled by recourse to philo-

3. Isaac Newton, "Preface to the Reader," *Mathematical Principles of Natural Philosophy*, trans. Andrew Motte, rev. Florian Cajori (Berkeley, 1934) xvii–xviii. Newton's profound and complex relation to ancient sources for all aspects of his thought is most comprehensively tackled by Rob Iliffe, *Isaac Newton: Priest of Nature* (forthcoming). For Bacon, see Antonio Perez-Ramos, *Francis Bacon's Idea of Science and the Maker's Knowledge Tradition* (Oxford, 1988).

4. In this respect, Proclus' thought is treated insufficiently in Jacob Klein's otherwise comprehensive *Greek Mathematical Thought and the Origins of Algebra*, trans. Eva Baum (Cambridge, MA, 1969). For a partial study of the fortunes of Proclus' *Commentary* in the early modern period see Giovanni Crapulli, *Mathesis Universalis: Genesi di una idea nel XVI seculo* (Rome, 1969). See also Neal Gilbert, *Renaissance Concepts of Method* (New York, 1960) 87–92, esp. note 33.

5. Nicholas Hartmann, *Das Proclus Diadochus philosophische Anfangsgründe der Mathematik nach den ersten Zwei Büchern des Euklidkommentars*, in *Philosophische Arbeiten*, herausgegeben von Hermann Cohen und Paul Natorp, IV. Band, 1. Heft (Giessen, 1909). See also Stanislas Breton's "Note liminaire" to the French translation of Hartmann's work, in Breton, *Philosophie et Mathématique chez Proclus* (Paris, 1969) 175–79. Both in *Die Platonische Renaissance* and *Das Erkenntisproblem*, Cassirer is generally silent on Proclus, apparently subsuming him under Plotinus.

6. The importance of Bacon's *Novum Organon* for Barrow's early natural philosophical thinking is seen in his oratorical compositions concerning Cartesian natural philosophy. In those, however, Bacon's views are synthesized with Platonic and Stoic sources. A discussion and full translation of the most important of these orations is found in Ian Stewart, "Fleshy Books: Isaac Barrow's Oratorical Critique of Cartesian Natural Philosophy," *History of Universities* 16 (2000): 35–102.

sophical authorities. He seemed thereby to relegate philosophy to the sidelines of mathematics, making it only a handmaid to mathematical thinking. In doing so Barrow explicitly (and understandably) opposed the *Commentary* of Proclus. On the other hand, not only in choosing which questions to tackle concerning the nature of mathematics, but even in his treatment of those questions, Barrow relied, in part, on Proclan formulations. This is particularly true of the central question of whether and how mathematics brings into relation the domains of the senses and the intellect. With respect to this question, Barrow "overcame" Proclus by assuming, in part, Proclus' position. Barrow did so in response to a more immediate, contemporary scholarly debate that originated partly in Proclus' *Commentary*. In seeking to balance two opposing directions of that debate, Barrow in fact returned to at least one aspect of Proclus' original position, even as he sought to oppose him in a quite general way. Insofar as Barrow's thinking was representative of an early modern account of how mathematics belonged to—and even defined—philosophy, this ambiguity suggests the need to reconsider Proclus in the early modern period. We must attend to how Proclus' texts were actually read and used in the formulation of early modern positions, even when those positions are stated as oppositions to Proclus' thought.

PROCLUS AND BARROW: GENRE AND FORM

Barrow's introductory lectures on mathematics were given during the first three years of his tenure as first Lucasian Professor at Cambridge (1663–1669). They were published posthumously in 1683 as twenty-three lectures, the *Lectiones Mathematicae*. In his remaining time as Lucasian Professor, Barrow produced two further sets of lectures, in which he established himself as a highly respected mathematician in the fields of curvilinear geometry and geometrical optics.[7] Barrow also published a very popular edition of Euclid's *Elements of Geometry*, and produced authoritative Latin editions of more challenging Greek mathematicians such as Apollonius and Archimedes.[8] Barrow's reputation as an English flower of scientific, but also theological, scholarship secured for his *Lectiones* a noteworthy audience. One was the

7. *Lectiones Mathematicae XXIII; in quibus Principia Matheseos generalia exponuntur: Habitae Cantabrigiae A.D. 1664, 1665, 1666* (London, 1683), hereafter cited as *LM* and *Lectiones*. The *LM* were translated in 1734 by Rev. John Kirkby under the title *The Usefulness of Mathematical Learning Explained and Demonstrated.* William Whewell collected in one volume the Latin texts of all Barrow's lectures, *The Mathematical Works of Isaac Barrow, D.D.* (Cambridge, 1860). I shall, in referring to the *LM*, use the pagination from Whewell's text. The translations given from the *LM* are my own.

8. For a summary of Barrow's life and work, see Ian Stewart, "Isaac Barrow," in *The Dictionary of Seventeenth-Century British Philosophers*, ed. Andrew Pyle (Bristol, 2000) 68–74.

undergraduate Isaac Newton, who likely sat in Barrow's audience at Cambridge; others included Leibniz, Hume, and Berkeley.[9]

The twenty-three lectures that make up the *Lectiones* can be organized into four sections, each of which treats *questiones* that belonged to contemporary literature in the foundations of mathematics. The lectures as a whole take the form of a steady march through a range of these *questiones.* The first section (lectures I–III) deals with the designation or name, subject matter, and internal divisions within the discipline. The second (IV–VIII) treats the means of demonstrating conclusions regarding the predicates of mathematical objects. Section three (IX–XVI) explores the predicates themselves: equality and inequality, geometrical space, termination, divisibility, and comparability. The final section (XVII–XXIII) continues the discussion of comparability, or proportion theory, with particular emphasis on defending Euclid's approach to proportion theory in the *Elements.* As the overall title of their published version makes clear, the lectures were intended to expound the "general principles of mathematics," not so much teaching the student how to *do* mathematics, but rather explaining mathematics as a body of learning and as a mode of reasoning, fixing its place and role among the sciences.[10]

As such, the *Lectiones* stood in a tradition of commentary on Euclid's *Elements* that relied upon Proclus' *Commentary* as paradigm. Already at the quite general level of *genre,* Barrow's *Lectiones* stand in ambiguous relation to Proclus simply because the period itself did. On the one hand, Proclus was key to explaining to Renaissance scholars why mathematics was proper training for an ascent to philosophy and theology.[11] Proclus' *Commentary* explained why Euclid's *Elements,* and mathematics generally, could provide a way out of the sterile linguistic disputes of the "schoolmen." Hence Proclus

9. Barrow's reputation as a scholar and mathematician amongst Continental and British thinkers is surveyed by Mordechai Feingold, "Newton, Leibniz, and Barrow Too: An Attempt at a Reinterpretation," *Isis* 84 (1993): 310–38. Berkeley's reading of the *Lectiones* was extensive and crucial for his position, especially in how he departed from Barrow; see Douglas Jesseph, *Berkeley's Philosophy of Mathematics* (Chicago, 1993) *passim.* Barrow's influence on Newton has received frequent comment, most recently by Peter Dear, *Discipline and Experience: The Mathematical Way in the Scientific Revolution* (Chicago, 1995) ch. 8.

10. The contents of the *LM* have been surveyed by Mahoney, "Barrow's Mathematics: Between Ancients and Moderns," in *Before Newton,* ed. Mordechai Feingold (Cambridge, 1990) 179–201.

11. For example, Christopher Clavius' *Prolegomena* to his famous edition of Euclid's *Elements*: "Demonstrant enim omnia, de quibus suscipiunt disputationem, firmissimis rationibus, confirmantque, ita ut vere scientiam in auditoris animo gignant, omnemque prorsus dubitationem tollant." *Commentaria in Euclidis Elementorum Libri XV,* 2nd ed. (Rome, 1589) 3–5. Similar views as expressed by the Englishman Thomas Digges are discussed by Stephen Johnston, "Making Mathematical Practice: Gentlemen, Practitioners and Artisans in Elizabethan England" (Ph.D. dissertation, Cambridge University, 1994) 76–83.

was of interest to the humanist reform of the universities.[12] Proclus provided a way of reading Euclid's use, for example, of 'axiom,' 'definition,' and 'postulate' in a way that illumined what it was to learn and to teach mathematics. Not only was Proclus a key source of information concerning the history of mathematics, and an expositor of Euclid, he was also regarded as an aid to comprehending both Plato's and Aristotle's treatment of mathematics as one of the theoretical sciences.[13] Moreover, the notion of an ascent from the sensible to the intelligible expounded in the *Commentary* was one still assumed by the internal hierarchy of the early modern university. It informed the whole fabric of its moral and religious discipline. The frequent argument that mathematics in particular could aid the mortification of the passions needed for scholarly study owed its force to the logic of the turn inward from the world to the quietude of the intellect. This was a logic the early modern period found in Hellenistic sources generally, and particularly in Neoplatonism. In Barrow's Cambridge, one finds this logic preached from college pulpits and hammered home in tutors' 'advice manuals,' which routinely appealed, *inter alia*, to the intelligible nature of mathematical study (and scholarly study generally) as an antidote to the fleeting and tumultuous experiences of the senses.[14]

All this notwithstanding, Proclus was for Barrow still a *commentator*: an authority in scholastic dispute. He had to be overcome in order to return to the pure ancient source of Euclid's text—*ad fontes*—in the spirit of two centuries of Renaissance (and also Reformation) scholarship which sought to do away with an interposing scholasticism that had sullied the ancient texts

12. These reforms and their appeal to mathematics are reviewed in Ian Stewart, "Authorized Reason and Reasonable Authority" (PhD dissertation, Cambridge University, 1998) ch. 5. Crombie, *Styles of Scientific Thinking* 41; *idem*, "Science and the Arts in the Renaissance: The Search for Truth and Certainty, Old and New," *History of Science* xviii (1980); *idem*, "Mathematics and Platonism in the Sixteenth-century Italian Universities and in Jesuit Educational Policy," in *Prismata: Naturwissenschaftsgeschichtliche Studien: Festschrift für Willy Hartner*, ed. Y. Maeyama und W.G. Saltzer (Wiesbaden, 1977) 63–94.

13. The key texts here are Plato's *Republic* 509d–511d, and Aristotle's *Metaphysics* 1026a18–19, 1064b1–3. See Clavius, *Commentaria*; Francisco Barozzi, *Opusculum ... de certitudine & ... de medietate Mathematicarum continentu* (Padua, 1560) 40; Josephus Blancanus, *De Mathematicarum natura dissertatio* (Bologna, 1615) 27.

14. A typical expression of this is Humphrey Prideaux's commencement sermon at Oxford during the 1620s: "Scala visibilium ad Invisibilium," in *Viginti-duae Lectiones de totidem Religionis Capitibus* (Oxford, 1648) 54–64. Barrow refers to it in his inaugural lecture: Alexander Napier, ed., *The Theological Works of Isaac Barrow*, vol. 9 (Cambridge, 1859) 211. See also Benjamin Whichcote, *Several Discourses*, ed. John Jeffery, vol. 2 (London, 1701–1707) 400; John Smith, *Discourse demonstrating the Immortality of the Soul* ch. 5, in *Select Discourses*, ed. John Worthington (London, 1660); Aharaon Lichenstein, *Henry More: The Rational Theology of a Cambridge Platonist* (Cambridge, MA, 1962) 28 ff.

themselves.[15] In the course of his lectures Barrow frequently turns to Euclid's language as being clearer than subsequent attempts (including Proclus') to explain it. For example, treating the notion of 'limit' in geometry, he remarks concerning Proclus' lengthy discussion, that as soon as philosophers try to make such things clear in philosophical language, they cease to be understood, or even to understand themselves: "etiam subtillisimi Philosophi vix consona dicere possunt, aut alii aliis, aut sibimet iidem ipsius" (*LM* 131).[16] In regard to language in general, Barrow's attitude to Proclus "and those who followed him" is that the more ancient texts are more clearly expressed, and in language common to all disciplines and to the common notions of learned discourse.[17]

Beyond the linguistic argument, the whole activity of Proclus' *Commentary* to render a philosophical account of Euclid's text seemed, in a sense, unnecessary. Barrow's *Lectiones* constitute a well-known statement of a widespread early modern development that looked to the sciences themselves for their *own* philosophical foundations.[18] This was true for Barrow even when, ironically, he found it hard to free himself from philosophy's authority over mathematics. For example, by his thirteenth lecture Barrow had only just reached arguments in the literature over Euclid's axiom of equality by congruity (*Elements* Bk. I, Axiom 5). Accordingly, in a long *apostrophe*, Barrow asked his auditors rhetorically: "shall I never extricate myself from these quirks and trifles?" After all, debates in philosophical literature regarding the history of mathematics, or concerning definitions, the nature of number, or equality

15. Anthony Grafton, "Barrow as Scholar," in *Before Newton* 291–302. For an example of Barrow's respect for ancient mathematical texts, see his *Archimides Opera: Apollonii Pergaei Conicorum libri IIII. Theodosii Sphaerica: Methodo Nova Illustrata, & Succincte Demonstrata* 3 pts. (London, 1675, 4°) epistola lectori.

16. He refers his readers to Descartes' *Principiae Philosophiae*; *Oeuvres de Descartes*, vol. VIIIA, ed. C. Adam and P. Tannery (Paris, 1964–1976) 8. He could, however, equally prefer Aristotle over Descartes for the same reason; *Oratio Sarcasmica in Schola Graeca* (1661), in T.S. Hughes, ed., *The Works of Dr. Isaac Barrow*, vol 6 (London, 1830) 385.

17. In lecture XVII, for the case of Euclid's definition of 'ratio' (*Elements* Book V, def. 3), Barrow canvased the use of the term in Greek *non*-mathematical contexts in order to show that the sense of the word as used by Euclid was, by contrast to later commentators, "not far from the common use that writers make of it in other disciplines, nor is it repugnant to the common way of speaking" (*LM* 271).

18. Gary Hatfield, "Metaphysics and the New Science," in *Reappraisals of the Scientific Revolution*, ed. D. Lindberg and R. Westman (Cambridge, 1990) 93–166. Derek Whiteside, "Patterns of Mathematical Thought in the Later 17th Century," *Archives for the History of the Exact Sciences* 1 (1960): 179–80. Paulo Mancosu, "Aristotelian Logic and Euclidean Mathematics: Seventeenth-century Developments of the *Questio de certitudine mathematicarum*," *Studies in History and Philosophy of Science* 23 (1991): 241–64. Amos Funkenstein, *Theology and the Scientific Imagination from the Middle Ages to the Seventeenth Century* (Princeton, 1986).

> scarcely touch the outmost skin of geometry, much less reach its inner recesses .... While disagreements and strife, shouts and disturbances make a racket, in its secluded parts, a deep peace and profound silence dwells within the walls, and inside the fortress itself there is no controversy or opposition. (*LM* 205)

Barrow's solution is to remind his audience that the truth, clarity, and certainty of mathematics were not at issue, only those questions such as the "order of propositions" and the "method and mode of knowing." These, he says revealingly, have more to do with "philosophical, external things" than with mathematics. Hence he refers to his treatment of these "*generalia*" as more suited to the "fleeing words" and "unfaithful caverns of the ears" of the spoken lecture. By contrast, mathematical problems and theorems require things to be placed before "faithful eyes" (*LM* 208–09; 213). What he meant was the actual *doing* of mathematics, in this case especially the turn to Euclid's text. According to Barrow, that turn brought a "deep peace and profound silence" to the tradition of noisy debate inaugurated by Proclus' *Commentary*.

But what characterizes the consistent thrust of the *Lectiones* is *not* that the debates could be put aside. In fact, they could be *resolved* by such a turn. These resolutions are inescapably philosophical, as Barrow himself knew, his quips about philosophy notwithstanding. And here we find at the general level of Barrow's overall approach a fundamental ambiguity in his relation to Proclus. Barrow turned to mathematics for a philosophical resolution to queries coming from "outside it." But this turn to mathematics itself is central to Proclus' *Commentary*, although he would never have defined such questions as "outside" mathematics. Proclus' task was to expound in a way faithful to Plato's dialogues the intermediary—and mediating—character of mathematics relative to what belonged lower and higher than itself, both with respect to being and to knowing.[19] Characterizing mathematics as this intermediary already determines the fundamental question concerning it, namely how mathematics could both have its own subject matter and methods proper to it, and also be propaedeutic for what is above it and paradigmatic for what is below it—the intelligible and the sensible, respectively.[20]

19. This goal is announced already in the opening chapter of the prologue: *Commentary on the First Book of Euclid's Elements*, vol. 3, trans. Glenn R. Morrow, 2nd ed. (Princeton, 1992) 1–5 (all subsequent references to the *Commentary* are from this translation, and will be cited by reference to the standard Friedlein pagination). This fundamental goal structures Charles-Saget's entire study of Proclus relative to Plato and Plotinus, *L'Architecture du Divin. Mathématique et Philosophie chez Plotin et Proclus* (Paris, 1982). See also Morrow, trans., *Commentary* xxxii ff.

20. Following Charles-Saget, I take this question and its careful treatment in Proclus as what marks his development of Plato's thought; cf. especially *L'Architecture* 193. Proclus did not of course present his thinking as distinguished from Plato's, and he explicitly defends Plato from the accusation that he criticized mathematics for "not knowing its starting points"; *Commentary* 29.15–32.20.

The task is, in other words, to think through a logic of mediation, while affirming nevertheless the fundamental distinction of mathematics as a thinking that does *not* know the "intermediary condition between being and nonbeing," and which resists a self-discovery of its relation to the intelligible.[21]

But Proclus insists that this resistance must be overcome, and that it belongs to mathematical thinking to do so. In the *Commentary* he wastes no time initiating the reader. Already in the opening pages of his prologue he states: "to find the principles of mathematical being as a whole, we must ascend to those all-pervading principles that generate everything from themselves: namely the Limit and the Unlimited." He proceeds immediately to Euclid's mathematics, relying on a rather difficult aspect of his treatment of commensurable and incommensurable magnitudes (*Elements* X, def. III and IV). "For number," says Proclus, "beginning in unity, is capable of indefinite increase, yet any number you choose is finite .... If there were no infinity, all magnitudes would be commensurable and there would be nothing inexpressible or irrational, features that are thought to distinguish geometry from arithmetic."[22]

We have in Proclus then the demand for a philosophical reading of mathematical thinking that is both properly mathematical as well as transcending it. This general feature of the *Commentary* makes it difficult for Barrow, even at the level of the genre and overall approach of his *Lectiones*, to carry out an unambiguous departure from Proclus merely by appeal to the self-sufficiency of mathematics. Proclus' careful attention to mathematical thinking, as it appears in Euclid's *Elements*, constituted such a substantial clarification of Euclid precisely because Proclus understood such attention to be requisite to a movement beyond mathematics to its foundations. Proclus' *Commentary* consists not simply of explications of Plato's texts, such as those found in the prologue, but—substantially—of expositions of Euclid's texts.[23] His account of propositions and constructions was no less part of the *Commentary* than

21. The resistance is overcome only by a kind of conversion; Charles-Saget, *l'Architecture* 194–201. She locates in this resistance of mathematics Plato's fundamental insight, one to which Proclus remains faithful, but which constitutes the difficulty of articulating how mathematical thinking could be mediatory; *L'Architecture* 13.

22. *Commentary* 5.13–17, 6.13–20. On this general feature Charles-Saget comments: "Cela signifie qu'en posant la nécessité d'une conversion proprement mathématique, Proclus entreprend de surmonter la discontinuité que marquait chez Platon la divergence des deux attitudes, philosophique et mathématique. Et ce n'est pas ici le philosophe qui accomplit la tâche du mathématicien: c'est le mathématicien lui-même. Car c'est en tant que mathématicien que ce dernier est sommé de revenir aux principes de son savoir" (195).

23. The acumen of Proclus' reading of book I of the *Elements* is attested to throughout Sir Thomas Heath's definitive translation and critical edition of the *Elements*: *Euclid, The Thirteen Books*, 2nd ed., vol. 1 (New York, 1956) *passim*. Heath had little patience for Proclus' transcending strictly mathematical discussion.

were his discussions of the Platonic ascent. Proclus treated the *Elements* as subject to philosophical commentary *qua* mathematics because mathematical thinking belongs to *dianoia*. Though having its ultimate source in *nous*, mathematical thinking at the level of Soul is also produced from within Soul itself.[24]

It was by reason of this double focus that Proclus' *Commentary* was such a rich source of philosophical reflection for early modern mathematicians and philosophers alike, to the extent that it could produce a tradition of debate that still provided the starting point for most of Barrow's *Lectiones*. It is true that Barrow, by insisting on a turn to the "inner peace" of Euclid himself, explicitly opposed Proclan moves to transcendence as "external" to mathematics. The generation of mathematics from within *noiesis* and ultimately from the One, is the side of Proclus to which Barrow's thinking remains explicitly antithetical. But Barrow's own philosophical resolutions of such "externals" rendered those resolutions *internal* to mathematics precisely by his turn to mathematics, and to the text of Euclid's *Elements*. This is, in overall approach, the other side of Proclus' *Commentary*. And central to that side of the *Commentary* is Proclus' account of the relation of sense and intellect in mathematics. It is then not surprising that on this fundamental question of the relation of sense to intellect in mathematics, Barrow too will depend on Proclus—even in trying to depart from him.

PROCLUS ON MATHEMATICAL BEING AND KNOWING

For Proclus, mathematical thinking stands in relation to the sensible as paradigm. In a central passage in which Proclus interprets Plato's image of the line in the *Republic* 509d–511d, he draws a relation between the way in which "picture thought" (the lowest on the line) is a likeness of a likeness, and the way understanding (*dianoia*) "studies the likenesses of intelligibles." This relation of the two orders of knowing—through their both being a kind of likeness—accounts, according to Proclus, for the mediating character of mathematics: "therefore," he continues:

> mathematical objects have the status neither of what is partless and exempt from all division and diversity nor of what is apprehended by perception and is highly changeable and in every way divisible .... He [Plato] shows that conjecture [picture-thought] has the relation to perception that understanding has to intellection. (*Commentary* 11.3–5, 11–14)

24. For example, Proclus' notion of the soul's mathematical thinking as "self-moving"; *Commentary* 15.16–16.4; see also 54.15–55.6: though the "circle in the understanding is one, yet geometry speaks of many circles" (15.20–21). See also Gregory MacIsaac, "The Soul and Discursive Reason in the Philosophy of Proclus" (PhD Dissertation, University of Notre Dame, 2001) ch. 2, n. 41. I am grateful to him for making his dissertation available to me.

Articulating that mediation requires of Proclus that he first rebut the view that mathematical thought has its rise in sense perception. He asks rhetorically:

> Should we admit that they are derived from sense objects, either by abstraction, as is commonly said, or by collection from particulars to one common definition? Or should we rather assign to them an existence prior to sense objects, as Plato demands and as the processional order of things indicates?[25]

The treatment of this question leads Proclus finally to an account of Soul, which "contains in advance all mathematical concepts, since it is their originating principle. By virtue of its power it projects from these previously known starting-points the varied body of mathematical theorems."[26]

But having rebutted an account of the origin of mathematicals as arising from abstraction from sense particulars, Proclus' association of mathematics with Soul (and later, imagination) immediately recovers the connection of mathematicals to the objects of sense:

> Because it is subordinate to the principles of the One and the Many, the Limit and the Unlimited, the objects under its apprehension occupy a middle station between the indivisible forms and the things that are through and through divisible .... The range of thinking extends from on high all the way down to conclusions in the sense world, where it touches on nature and cooperates with physics in establishing many of its propositions, just as it rises up from below and nearly joins intellect in apprehending primary principles.[27]

Not only is Proclus meeting the demand of Plato's *Timaeus* that mathematical forms are involved in the construction of the universe, but he is also meeting the demand that mathematics afford the occasion for rising from

25. The criticism of abstraction from sense objects as an account of mathematical objects extends from 12.2–15.15, and comprises a tripartite attack, neatly summarized by MacIsaac, "The Soul and Discursive Reason" ch. 2, sect. 1.a.1–3; J. Trouillard, *L'Un et l'âme selon Proclos* (Paris, 1972) 38–50.

26. The discussion is a long one, extending from *Commentary* 12.2–18.6, framed by the above citations.

27. *Commentary* 19.10–25. He mentions mixed mathematical sciences, in particular mechanics, optics, and catoptrics. On the identification of the domain of Soul with the domain of mathematics, see Philip Merlan, *From Platonism to Neoplatonism*, 2nd ed. (The Hague, 1960) 11–33. This account is questioned by Ian Mueller, but without argument; see Morrow, *Proclus*, 2nd ed. (Princeton, 1990) xix, n. 31. MacIsaac has articulated a convincing double sense of Soul's dianoietic activity, depending on whether through it the Soul is regarding *Nous* or Body. In the first case, *dianoia* is dialectic, and in the second it is mathematics; "The Soul and Discursive Reason" ch. 4, sect. 1.

the sensible to the intelligible. Mathematics belongs neither to intellect, which is "steadfastly based on itself," nor to opinion and perception, which "fix their attention on external things and concern themselves with objects whose causes they do not possess." Mathematics, "though beginning with reminders from the outside world, ends with ideas that it has within; it is awakened to activity by lower realities, but its destination is the higher being of forms" (18.10–20). His account of soul as intermediary is critical to both the epistemic and the cosmological role of mathematics as mediation. Otherwise there could be no 'reminder' from the sensible realm. This comes out most clearly in the second part of his prologue, concerning the imagination.

Proclus' account of imagination begins with the recognition of the difficulty in accounting for the being of geometricals. If we would say the figures of the geometer are "belonging to the sense world and inseparable from matter," then how can we speak with Plato of an emancipation from sensible things, "in preparation for activity in accordance with Nous"? But if we would assert that the objects of geometry are "outside matter, its ideas pure and separate from sense objects," then "none of them will have any parts or body or magnitude." Yet, in a sense inherent to mathematical thinking itself, they clearly do have such "parts or body or magnitude." What is needed is a "receptacle that accommodates indivisibles as divisible, unextended things as extended, and motionless things as moving."

Proclus thus seeks an account that agrees "both with the facts themselves" of the science of geometry, and with Plato's demand that mathematics belong to the *dianoeta* which separate us "from sensible things, and incite us to turn from sensation to Nous." And this, as he regards it, is an improvement on "what Porphyry ... and most of the Platonists have set forth," in taking more complete account of both the practice of mathematicians and the thought of Plato (49.4–50.10; 56.22–57.9). In an extended discussion intended to meet both demands (50.13–55.6), Proclus distinguishes between two kinds of universals and their corresponding individuals: those found respectively in the imagination and those found in sense particulars (53.18ff). Proclus attempts both to relate and to distinguish the way in which particular sensible geometricals participate in their universals, and the way in which geometricals as found in imagination participate in their universals. His account concerns how the intelligible matter of the imagination takes on *analogous* aspects of particularity to that experienced in sensible examples of geometricals. This whole account explicates how the imagination that studies "likenesses of intelligibles" is analogous to the "likenesses of likenesses" of the sensory realm.

I cannot do justice here to the enormous importance of this notion of *analogia* for Proclus' system, or for how it clarifies his treatment of the Aristotelian notion of "passive nous" and "intelligible matter."[28] Proclus' *Commentary* was relevant for Barrow's *Lectiones* because this account of *analogia* showed how geometrical thinking had both a particular and a universal character, or equivalently, a sensible and an intelligible character, without being identified with either sensation or intellect alone. This account was influential in the period preceding Barrow, and was one to which he responded directly. What Proclus attempted to hold together in fact produced diverging readings that emphasized one side or the other. It is to these one-sided accounts of mathematics that Barrow is responding in his *Lectiones*, and whose respective demands he seeks both to meet and to balance. In that light, Barrow's ambiguous relation to Proclus becomes clearer: in opposing Proclus, Barrow in fact restores one aspect at least of Proclus' more unified position, even while transforming it fundamentally.

PROCLUS IN BARROW'S *LECTIONES*

Barrow's first lecture begins by considering the meaning of the word *mathesis*, moving then to the larger question of whether or not mathematics is a theoretical science, and what rank it enjoyed among the sciences.[29] He at once challenges contemporary reasons for mathematics' preeminent status. In doing so he opposes immediately and explicitly the account of Proclus, and any account derived from him which might represent mathematical thought as more intelligible than sciences dealing with sensible experience.[30]

Barrow explicitly cites Proclus' tripartite view of the soul and of the modes of cognition. As Barrow represents it, Proclus' account of the soul belongs to the mediating role of mathematical study, drawing the student away from the world of the senses and turning inward to contemplate the ideas. As Barrow renders Proclus' view, mathematics has that office most properly to "promote notions of things, arouse contemplation, purge the mind, draw forth innate ideas, draw away ignorance and forgetfulness, and dissolve the chains of error" (*LM* 25–6). Although Barrow introduces this account of the

28. It is fully discussed by MacIsaac, "The Soul and Discursive Reason" ch. 4, sect. 3–4. The full account of *analogia* is worked out by Proclus only in his commentary on the *Timaeus*, but its fundamental result is assumed in the Euclid commentary, and is crucial to the mediating character of both World Soul and individual soul.

29. *Mathesis* was the root of 'mathematics,' but was translated *disciplinas*, and Barrow repeats the common invocation of Aristotle's *Posterior Analytics* 70b9–72b4; *LM* 24.

30. A review of this notion of *mathesis* and its influence on sixteenth-century treatises is given by Crapuli, *Mathesis Universalis*; see also Guenther Risse, *Die Logik der Neuzeit*, vol. 2 (Stuttgart, 1970) 32 ff. Worth noting here is Leibniz's teacher, Erhardus Weigelius, whose *Analysis Aristotelica ex Euclide restituta* (Jena, 1658) argued that mathematics was not only the study of quantity, but also that "mathesis non sit pars philosophiae ... sed quod ipsissima philosophia, tota, recta, ratiocinans" (Risse, 143, n.596).

name of *mathesis* "because many deride it before having understood it," he nevertheless dismisses it, replacing it with an historical argument for the pre-eminence of mathematics within the ancient academies. "All of these," says Barrow in reference to Proclus, "are rather more elegantly produced than truly asserted."[31]

Barrow next moves in ordered fashion from discussing the name of mathematics, *mathesis*, to considering the 'object' of mathematics. This raises the question of the nature of abstraction, or of how 'pure' mathematical objects are distinct from those of mixed mathematics. He is careful to define abstraction as a movement from a particular to a universal, and to point out that the effect of such a definition is to give an account of abstraction in mathematics that is "no different" than in other sciences, such as physics, contrary to "whatever some may strangely say." Taking his cue from a passage in Aristotle's *Metaphysics* that subordinates the mixed sciences—such as optics—to geometry, Barrow says there is likewise in physics a hierarchy of abstraction. Beginning first with the

> constitutive principles of body taken universally, it [natural philosophy] inquires into matter and form, and such like, then it pursues the common affections of any body (quantity, place, motion, rest and such like); then it descends to the next species, and investigates likewise the special natures and properties of them.

Barrow argues that this form of abstraction is like that in mathematics, by which one understands objects to be more or less 'abstract' depending on the generality of the terms used:

> magnitude in general with its general affections of divisibility, congruence, proportionality, capacity of different situation and position, mobility and others, declaring these to be in magnitude, and in what manner. Lines, planes and surfaces constitute its species, each of which have specific properties, which can be further subdivided until one arrives at the lowest division possible, all rules and theorems having been demonstrated by proper reasoning.

He treats the question of how mathematical objects are abstracted from sensible physical ones, and hence the nature of mathematical universals, as requiring no great consideration. As Barrow puts it, by considering, for exam-

31. Barrow argues that the term 'mathematics' or *mathesis* was by common currency associated with studies propaedeutic to philosophy, since the other disciplines of the trivium had not been formalized into teachable subjects. In the historical account he seems to follow aspects of Vossius' account in his encyclopedic *De Universeae Mathesios natura & Constitutione Liber* (Amsterdam, 1650) ch.1, sect. 4. The view of the ancient Greek as lacking a developed sense of the trivium was an early modern humanist one, articulated already in the fifteenth century. See Sarah Stever Grevelle, "The Latin-Vernacular Question and Humanist Theory of Language and Culture," *Journal of the History of Ideas* 49 (1988): 367–86.

ple, the 'universal' distance from the sun to the earth, one has achieved mathematical abstraction. He thus treats that distance unspecified by a number, and represented by a geometric line—rather than one of an actual dimension found in the world—as being a mathematical line rather than a physical one. To emphasize the point, further on he says: "Just like the physicist, geometry begins by setting before itself magnitude taken universally, and not one peculiar to this or that body, a magnitude of the heavens, or of the earth, or of the sea."[32]

This account of abstraction in effect effaces the medieval distinction between *abstractio totius* and *abstractio formae*.[33] The first referred to the move from more specific to more general terms of a science, such as from 'a man' to 'man' to 'animal.' The second referred to the mathematical abstraction from physical bodies by a 'cutting away' of sensible qualities. The first is participated in by all the sciences; the second belongs properly to mathematics. That distinction requires the distinction between 'intelligible matter' and 'sensible matter.' In effacing the distinction between the two kinds of abstraction, Barrow is bypassing the whole Proclan doctrine of distinct but analogically related domains of universals in the imagination and in sensation. This in turn was central to Proclus' tripartite account of the self—and accordingly of learning—that Barrow had dismissed earlier in his treatment of *mathesis.* Barrow's opposition to the Proclan system is both immediate and self-consciously systematic to the point of directing most of the *Lectiones* as a whole.

Consistent with the direction of his first lecture, in the second Barrow takes up the traditional distinction between pure and mixed mathematical objects, in order to abolish a distinction between a *scientia* limited to the objects of pure mathematics, and some lower form of knowing reserved for sensible objects. He first runs through ancient distinctions from the Pythagoreans to Geminus. Citing Proclus, Barrow renders Geminus' view thus: "they thought the mathematician treated on the one hand things to be perceived alone by the intellect [*res intellectu solo percipiendas*], and on the

32. *LM* 31–33; The Aristotle passage is *Metaphysics* 1061a.29ff. In that part of the *Metaphysics,* the concern is to show how a science of 'being' can be possible; or as Aristotle says: "how there can be one science of things which are many and different in genus" (1061b16–17).

33. This has also been noted by Jesseph, *Berkeley's Philosophy* 14–16. Although he does not cite his sources, Barrow shows that he is clearly aware of the medieval distinction by the way he deliberately identifies both sides, and treats them as equivalent. The distinction can be found in Thomas Aquinas, *Super Boetium De Trinitate, Opera Omnia Leonina,* ed. Fratrum Praedicatorum, vol. L (Rome and Paris, 1992), Questio 5, art. 3, 149, lines 239–55. See Claude Lafleur, "Abstraction, séparation et tripartition de la philosophie théorétique: quelque éléments de l'arrière-fonts Farabien et Artien de Thomas d'Aquin, *Super Boetium 'De Trinitate'* question 5, article 3," *Recherches de Théologie et Philosophie Médiévales* 67.2 (2000): 249–69.

other, things open only to the senses [*res sensibus obnoxias*]" (*LM* 39). Barrow asserts, on the contrary, that:

> every mathematical object is both intelligible and sensible in a different respect. It is intelligible insofar as the mind is able to apprehend and behold the universal idea of it; it is sensible insofar as it is in and agrees with the individual objects that impinge on our senses. For who does not view with the eye and feel by hand the individual dimensions of bodies? There is no reason a science [*doctrina*] of generals should be distinct from the consideration of singulars, since the former includes and respects the latter as a primary intention. Does knowledge [*scientia*] treat the intelligible sphere differently than it does the sensible sphere? Since the two are really the same, and subordinated only by an act of the mind, nothing can rightly be attributed to an intelligible sphere (that is, understood universally) that cannot with the greatest right be accommodated to and agree with the sensible one (that is, to every particular sphere).[34]

The distinction challenged here is that between kinds of knowledge, depending on whether the object is 'intellectual' or 'sensible.' On this account, natural science arrogates the name of mathematics to itself, "for there is no natural science from which the consideration of magnitude is wholly excluded" (*LM* 40). For this reason, Barrow proceeds for the remainder of the lecture to run through the long list of 'concrete' or 'impure' mathematical sciences, in order to point out that mathematics' place in them is due to the presence of magnitude in all these 'concrete' sciences. The point is made by the length of his list: the discussion of material magnitudes is no less 'mathematics' than is the discussion of 'immaterial' magnitude. Barrow's formulation explicitly avoids calling the mixed mathematics "parts of mathematics," thereby replacing Proclus' exact formulation concerning the subordination of the mixed mathematics to pure mathematics and, in turn, pure mathematics to a *mathesis universalis* (*Commentary* 18.6–20.7).[35] Hence in Barrow's treatment, the question of the subordination of sciences one to another is given a treatment which 'flattens' any notion of hierarchy.

As I shall illustrate further on, the notion of 'flattening' is an appropriate image, because Barrow does not in fact do away with the hierarchy; he will reinstate it when maintaining a distinction between the intelligible and the sensible aspects of mathematical experience. For the moment, however, it is enough to note that he draws out the consequences of this 'flattened' position in his steady march through virtually all the *questiones* concerning the

34. *LM* 38. On the scholastic terminology of first and second intention, and its early modern usage in discussing mathematicals, see Jacob Klein, *Greek Mathematical Thought* 208. Barrow's ironic retreat to this medieval distinction cannot be commented on here, but is typical of aspects of the *Lectiones* as a whole. In referring to 'sphere,' Barrow is evidently speaking of the three-dimensional mathematical object.

35. For a discussion of similar departures from Proclus' wording in Viète and Descartes, see Klein, *Greek Mathematical Thought* 181.

principles of mathematical thinking as found in Euclid. Barrow will apply this position to his account of arithmetic versus geometry, opposing both Proclus and those contemporary mathematicians who sought to place arithmetic 'above' geometry as the more abstract and universal ground (lecture III). As a consequence, he will oppose the contemporary view of algebra as the even more universal and abstract *mathesis universalis*, and as the proper language of proportion theory (lectures III, XX–XXIII).[36] He will—again in opposition both to Proclus and to contemporary invocations of Proclus—oppose any hierarchical epistemological distinction between speculative and practical forms of demonstration, between axioms and hypotheses, or between theorems and problems. And he will account for the greater certainty, clarity, and distinctness of mathematics on the basis that its objects, in contrast to other kinds of enquiry, are present both to sense and to intellect. The objects of those other kinds of enquiry are "*a sensibus abjunctam*" (*LM* 67; see lectures V–VIII). He will treat the intelligibility of notions such as the infinite divisibility or extension of magnitudes as established "by the firmest arguments and the most evident experiences" in terms of actual geometrical practice, whilst affirming simultaneously the inability of the human mind to conceive the infinite, or even the indefinitely extended (lecture IX), leaving that rather for the capacity of the divine (*LM* 133). He will tackle the thorny philosophical question of the nature of space, but "in a poorer manner, more accommodated to the common sense than to metaphysical notions" (*LM* 148). But in developing a notion of space agreeable to geometrical procedures of actual, individual constructions as found in Euclid, he attempts to solve the vexed early modern philosophical problem of God's relation to space (lecture X).[37]

By means of that solution he will defend in lecture XI a class of Euclid's proofs of equality that depend on a method of superposition, or geometricals 'lying' on top of one another. This method was grounded in Euclid's fourth axiom of book I: "Things which coincide with one another are equal to one another."[38] In order to defend a fruitful contemporary method of demon-

36. Proclus' discussion of *mathesis universalis* (*Commentary* 18.6–20.7) was used to justify the status of algebra, associated by some early moderns with general proportion theory. See for example Adrian van Roomen, *Apologia pro Archimede* (Geneva, 1597) 1–16; Barrow's contemporary John Wallis used the term to claim the priority of algebra over geometry; *Mathesis Universalis, seu Opus Arithmeticam, Phililogice & Mathematice traditum* (Oxford, 1657).

37. See my "Authorized Reason and Reasonable Authority" chap. 7; E. Grant, *Much Ado About Nothing: Theories of Space and the Void from the Middle Ages to the Scientific Revolution* (Cambridge, 1981) ch. 8, esp. 232–34.

38. Heath, *Elements* I: 155; On Euclid's use of superposition and its place in ancient mathematics see Heath, *Elements* I: 247–49. Cf. also Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge, MA, 1981); R.J. Wagner, "Euclid's Intended Interpretation of Superposition," *Historia Mathematica* 10 (1983): 63–89.

stration (the method of 'indivisibles' pioneered by Cavillieri) that superimposed geometricals of entirely different shapes, he is careful to separate the space bounded by a figure from the quantity of that space that will be equated with another geometrical of different shape.

> Congruity ... by means of a common space is extended to all magnitudes, insofar as no magnitude is bound to one space, or any space to one magnitude, but the place abandoned [*derelictus locus*] by one can be occupied by another .... This mental congruity of magnitudes ... is not absurdly supposed by Geometricians, for by this congruity there is not an actual or real penetration [of figures] which is being asserted, but rather as it is abstracted by the mind ... it is a general and indefinite capacity of admitting such a body, which is the principal property of space .... By means of this space, insofar as it is united and identified with quantity, quantities are joined and shown to be equal. (*LM* 170–71)

But he does so without taking away the sensible character of such procedures of superposition—i.e., the clarity of their motion in the space on the writing surface before one.

> There is here no use of the ruler or compass, no labour of the arms or hands, but it is entirely a work, artifice, and machination of reason ... there is, I say, nothing mechanical, except insofar as every magnitude is wrapped in matter [cf. Lecture II], exposed to the senses, visible and palpable, such that as the mind judges, the hand can follow, and practice can seek to emulate contemplation. This only establishes the strength and dignity of geometrical demonstration, and does not weaken or lower it, but raises it higher, making manifest by the senses the reality and possibility of the assumed supposition [*sumptae suppositionis*], establishing the authority of reason by the witness of the senses. (*LM* 167–68)

In other words, he retains simultaneously a sensible, spatial measure of such quantity, while abstracting the space from quantity. That abstraction was intended to retain the simultaneously intelligible and sensible character of superposition, and is a direct consequence of his position on abstraction developed earlier in lecture II.

In all of these, Barrow relentlessly pushed the demand that we cannot separate both aspects of geometrical thinking: the sensible and the intelligible. And this demand had its correlative in his epistemological account of the origin of mathematics in our experience. Later in lecture VII, in summing up his efforts to breakdown an epistemological and ontological hierarchy of axioms relative to hypotheses and definitions (as he perceived it in Proclus), Barrow asserts:

> From what has been said may be known the original of all natural science, and the genuine method of reasoning even from the first fountains of knowledge; which is nearly thus: the mind, from the observation of the things brought before it, takes occasion of framing ideas corresponding to them. As soon as the mind clearly perceives that

> these ideas agree with the things that may exist, it affirms and supposes them to exist. Then appropriating words to them, it forms definitions, and from the consideration and comparison of these together it draws consequences and makes theorems, which being joined together into certain systems, compose particular sciences. (*LM* 115)

Passages such as these make Barrow's account of mathematics appear as a form of empiricism. He has been read this way, although not by later empiricists such as David Hume.[39] But Barrow intended this 'epistemological' account as a kind of conclusion to his reflection on the coincidental—and inseparably—sensible and intelligible character of mathematical thinking. This character does not sit well with his alleged empiricism.

He does indeed portray his position as avoiding the doctrine of Plato's *anamnesis*, "because the mind without these *prolepses*, common *ennoiai* is sufficiently furnished with a native faculty able to acquire what is necessary for establishing the principles and middles [of demonstration] in the above-mentioned way." But neither, continues Barrow, "is it necessary to follow Aristotle in asserting that the truth of principles is established alone by induction through the perpetual observation of singulars, since in order to establish a mathematical hypothesis, or to form a definition, or to articulate a principle, only a single experience is needed, as long as it is sufficiently clear and indubitable."[40]

Running through his whole *Lectiones* is a view of 'mathematical experience' which simultaneously links and distinguishes the intelligible and the sensible aspect of such experience. I have described that experience as simultaneously sensible and intelligible. But it is fundamental for Barrow that in geometry there is also a very clear distinction between the sensible and the intelligible. Maintaining both of these demands requires of him a position that is Proclan in character, if not in inspiration. To see how he gets to that Proclan position, however, we must first turn to the actual context of scholarly debate that Barrow explicitly joined, at the beginning of which stood Proclus' *Commentary*.

PROCLUS IN BARROW'S CONTEXT OF SCHOLARLY DEBATE

Barrow's account of Euclid's methods of superposition, referred to above, was formulated explicitly in response to contemporary discussions that re-

39. Accounts of Barrow as an empiricist can be found in Helena Pycior, "Mathematics and Philosophy: Wallis, Hobbes, Barrow and Berkeley," *Journal of the History of Ideas* 48 (1987); Peter Dear refers to Barrow as an "empiricist mathematician" (*Discipline and Experience* 29), although he more generally characterizes Barrow's position as Aristotelian.

40. *LM* 116–17; he refers to Aristotle's *Post. Anal.* II. 19. It should be noted that Barrow recognizes here that Aristotle elsewhere questions this dependence on "singulars." He nonetheless chooses to represent his own position as between Plato and Aristotle.

lied in part on Proclus' account of such methods. For Proclus, geometricals in one sense are—and in another sense are not— in motion. This is consistent with his account of geometry as mediating the realm of motion and motionlessness. For example, with reference to Euclid's postulate I.3, Proclus says: "But let us not think of this motion as bodily, but as imaginary, and admit not that things without parts move with bodily motions, but rather that they are subject to the ways of the imagination"(186.9–11).[41] Some Renaissance commentators, by appeal to one side of Proclus, had argued that the absence of any 'real motion' in geometrical demonstration (which is supposed to be 'without motion' in one sense), meant that geometry was excluded from causal *scientia* as determined by Aristotle.[42] Others, using the other side of Proclus' account, objected that such motion was all too real, inappropriate for a science of things "without motion."[43]

But these divergent Renaissance accounts of superposition are only consequences of a more general divergence in how Proclus was read regarding the mediating character of mathematics between sense and intellect. Barrow was not only well aware of such developments, but many of his positions in the *Lectiones* were taken up in response to them. A principle aspect of that debate concerned whether mathematical objects could be the subject of causal demonstrations involving a syllogism having a middle term that expressed the operative cause, whether efficient, material, formal, or final.

The Jesuit mathematician Christopher Clavius, in arguing for the place of mathematics in the philosophical curriculum of the *Collegium Romanum*, cited the authority of Proclus in asserting that mathematicals were separated from matter only in the intellect, but not from sensible objects of physics themselves.[44] He saw himself as consistent with both Proclus and Aristotle when, following a well-established medieval formulation, he ascribed to mathematics the middle status between metaphysics and natural science, according to whether and how its objects were "joined" to matter. The key texts of Aristotle on the abstracted character of mathematicals from matter were taken by Clavius to give an ontological and epistemological priority to mathematics over natural science. In "sensible fact" mathematicals were "in

41. See also *Commentary* 51.13–54.14 and 185.20–187.4, where he develops Aristotle's 'passive *nous*' of *De Anima* 430a24, consistent with his account of geometry as mediating the realm of motion and motionlessness.

42. Coimbra Aristotelians such as Pereyra, *De Communibus omnium rerum naturalium principiis et affectionibus libri quindecim* (Rome, 1576) 70, 116, 118, cited in Mancosu, *Philosophy of Mathematics* 218, n. 59.

43. Further seventeenth-century concerns with the method are reviewed in Mancosu, *Philosophy of Mathematics* 28–33, such as those of Jacques Peletier and Flussus Candala. Others skirted the issue by denying motion actually was involved for those quantities said to be equal. A similar standpoint is found in Blancanus, *Dissertatio* 24.

44. Dear, *Discipline and Experience* 37, n. 17.

matter." In thought mathematicals were separate, whereas the subject of physical science was "joined to sensible matter both in fact and in thought," and the objects of metaphysics were "separate from matter both in thought and in fact."[45]

This neat account of Clavius was further commented on by fellow Jesuit, Josephus Blancanus. In his *De mathematicarum natura dissertatio*, appended to his longer compendium *Aristotelis loca mathematica* (Bologna, 1615), he sought to show how mathematicals could be the subject of causal demonstration. He was in fact responding to a half century of debate concerning the status of mathematical demonstration.[46] To do so, he explicitly retained this framework of mathematics as mediating physics and metaphysics in the sense used by Clavius. Blancanus made the following distinction. Unlike the physicist and the metaphysician, who consider quantity "absolutely, insofar as it is quantity," where its properties are "divisibility, locatability, figurability etc.," the mathematician always considers "delimited" quantity, whether numbers or geometricals.

> So it is obvious that these properties, which the mathematician considers, emanate [*per emenationem*] from this quantity insofar as it is delimited, such as equality, inequality, such and such division, transfiguration, various proportions ... etc. Obviously, these properites do not flow from the intrinsic nature of quantity, for if it is taken to be unterminated, the aforementioned properties do not follow, as nothing taken to be like this [unterminated], is equal or unequal, etc., but when termination is added to quantity, these properties flow from that termination by emanation. So it is correct to say that the formal aspect [*formalis ratio*] of mathematical thinking is termination, and that its total adequate object is terminated quantity, insofar as it is terminated.

In stressing this delimitation as *formalis ratio*, Blancanus sought to establish mathematical objects as objects of causal demonstration—that is, he sought to meet in mathematics the demand of Aristotelian science that there be essential definitions.[47] At the same time, this account of the limit or termi-

45. Clavius, *Commentaria*, cited in Crombie, *Styles* I: 489–91. Clavius was in fact using, somewhat confusedly, the medieval distinction between 'abstractio' and 'separatio.' For its orgins in early medieval thought, see Alain de Libera, *La querelle des universaux. De Platon à la fin du Moyen Age* (Paris, 1996) 110–16; Lafleur, "Abstraction."

46. The *Dissertatio* has been translated by Gyula Klima in the appendix to Mancuso, *Philosophy of Mathematics.* The importance of Blancanus to early modern debates is extensively discussed both by Mancuso, ch. 1, and Dear, *passim.*

47. *Dissertatio* 3–5. Later in his *Dissertatio* he will summarize: "Demonstrationes Geometricae non constant ex propriis, & per se, non enim Geometra considerat essentiam Quantitatis, neque eius passiones, quatenus ab illius essentia manant, quare ex communibus quibusdam, & mere extrinsecis necesse est procedere. Respondeo ex dictis cap. 1. de materia intelligibili, & definitionibus Geometricis huic obiectioni abunde fieri satis. materia enim Geometricae non est quantitas secundum se, sed quatenus terminata, cuius totam essentiam ex definitionibus essentialibus Geometra cognoscit" (19).

nation that causes "by emanation" the mathematicals is clearly Proclan in origin, finding its articulation in the *Commentary*.[48] But in stressing that the objects of mathematics were in fact objects of demonstrative science, Blancanus immediately stressed also their separation from objects of sense. In doing so he strained at the Proclan formulation of the mediating character of mathematics that had figured in Clavius' account. Immediately after stressing the delimited character of the mathematicians' subject, Blancanus continues:

> But this [delimited quantity] is usually called intelligible matter, in contradistinction to sensible matter, which concerns the natural scientist, for the former is separated [*separatur*] by the intellect from the latter, and is perceived by the intellect alone.[49]

For Blancanus, mathematical entities exist in the mind of the 'Author of Nature' and in the human mind as *rerum typi*, the archetypes of natural bodies, existing *per se* and as the true beings, literally existing "not in things, but in the mind of the Author of Nature."

> For this reason we should hold that these geometrical entities which are perfect in all respects are *per se* and true beings; whereas natural as well as artificial figures, which exist in the nature of things, as they are not striven for *per se* by any efficient cause, are beings only *per accidens*, and are imperfect and false.[50]

His account influenced his reading of the relation of Proclus to Plato and Aristotle. In explaining Plato's place for mathematics below *noiesis* in the account of 'the line' in the *Republic*, and Plato's apparent aspersion cast on mathematics as that which "dreams about being," Blancanus cites equally Marsilio Ficino and Proclus.[51] But in doing so, Blancanus stressed the divisions of the soul to such a degree that they are seen to exist merely in opposition.[52] As we have seen, Proclus' treatment of this passage in the *Republic*

48. See, for example, Proclus, *Commentary* 5.13–7.12, especially 7.1–3 in reference to the Pythagoren two columns; see also *Commentary* 136.19–146.17.

49. "Atque haec est illa Quantitas, aquae dici solet materia intelligibilis, ad differentiam materiae sensibilis, quae ad Physicum spectat; illa enim ab hac per intellectum separatur, ac solo intellectu percipitur" (6). This is an even further confusion of the medieval distinction between *abstractio* and *separatio* than that of Clavius. Blancanus effectively treats mathematics as a *separatio*, which belonged properly only to metaphysics.

50. *Dissertation* 6, 7; he gives the example of the triangle drawn on a map, for example, which is not a true triangle, but literally a "false" one. The true is that which is *in idaea Divina*.

51. *Republic* 533b; *Dissertatio* 20–24. Both Blancanus and Ficino, as well as those who used the discussion in the *Republic* to criticize mathematical thinking, appeal to Proclus' discussion in the *Commentary* 25.13–32.21.

52. *Dissertatio* 21: "totam autem Philosophiam inibi partitur in tres partes, in Dialecticam, seu Theologiam, quam intellectui attribuit absque ullo suppositione, & discursu: hancque solam in cognitione [Klima suspects a misprint here for *cogitatio*] seu ratiocinatione collocat, &

explicitly drew an analogy, not a mere opposition, between the imagistic thinking of *dianoiea* and that of sensation of particulars. Instead, Blancanus read that same Proclan treatment as reinforcing his own account of mathematics as belonging *wholly* to the intelligible realm.[53]

Ultimately at stake is Blancanus' effort to oppose Allesandro Piccolomini, a "very recent philosopher" who "frequently states that the certitude of mathematics derives from its showing everything to the senses, i.e., from its truths being perceived by the senses." Blancanus will appeal to Aristotle's inclusion of mathematics with physics and theology as theoretical sciences. This inclusion by Aristotle meant for Blancanus that Piccolomini's view was "entirely false": the "subject matter of mathematics is entirely intelligible, but not sensible."[54] Piccolomini had in fact resurrected a debate concerning a question in the Aristotelian commentary tradition, namely whether mathematics was a *demonstratio potissima.* The debate concerned whether in mathematics both what is 'prior for us' and what is 'prior in nature' coincide, unlike the objects of natural science whose causes and natures are more hidden from us.[55]

In somewhat notorious opposition to this, Piccolomini denied that mathematics conformed to Aristotelian strictures of *demonstratio potissima*: the premises of mathematics are not necessary, nor is the major premise always a definition of an essence; the demonstrations do not specify causes, and the middle term of demonstration is not immediate. Basing himself explicitly on Proclus, Piccolomini instead accounted for the certainty of mathematics by reference to its constructive procedure in imagination:

> So Proclus derives from Plato the view that the mathematical things themselves, about which demonstrations are made, are neither sensible things altogether in a subject, nor entirely freed from it, but that these mathematical figures are formed in the imagination, the occasion being afforded by quantities found in sensible matter. Moreover the

propterea principia supponit. tertio tandem in opinionem, aque versatur circa res naturales, quae in imaginatione ab eo collocatur." By *imaginatio* Blancanus does not have Proclus' notion in mind, but rather the lowest division of Plato's line: "picture thought."

53. *Dissertatio* 23: "cogitatio enim dicitur quasi coagitatio mentis, quae idem est cum discursu, aut ratiocinatione: quare manifestum est horum Philosophorum authoritate ratiocinationem versari circa Mathematicas, imaginationem vero circa res physicas." The Proclan passage in the *Commentary* is 10.15–12.1.

54. *Dissertatio* 27. The passage from Aristotle Blancanus is relying on is *Metaphysics* 1026a18–19; 1064b1–3.

55. The various moves in that debate are discussed in N. Jardine, "Epistemology of the Sciences," in C.B. Schmitt et al., eds., *The Cambridge History of Renaissance Philosophy* (Cambridge, 1988) 689–94. It was a position widely attributed to Averroes; Jardine, "Epistemology" 693, n. 31. Further treatment, including a summary of Barrow's position, can be seen in Mancosu, *Philosophy of Mathematics* ch. 1.

intellect derives those universal principles from these quantities that are in the imagination.[56]

The ensuing debate caused by Piccolomini diverged into two sides in the later sixteenth century amongst Paduan philosophers. One side stressed, along with Blancanus, the possibility of essential definitions and causal demonstrations based entirely on the intelligible matter of mathematics.[57] The other side followed Piccolomini in denying that mathematics possessed the status of causal, demonstrative science because of the place of mathematics within imagination. They did allow, however, that mathematics enjoyed a unique certainty because it enjoyed the aid of the senses.[58]

Closer to Barrow's own period, the redoubtable Pierre Gassendi sided with Benito Pereira in order to argue that since "no science exists, and especially no Aristotelian science .... I conclude that whatever certainty there is in mathematics is related to appearances, and in no way related to the genuine causes of things."[59] Even closer to home was Thomas Hobbes, who entered the *certitudo* debate in 1660 by asserting that geometry was the *only* science of causal demonstrations.[60] His position, however, given his stark nominalism, in fact had more in common with Piccolomini's reading of Proclus than with Blancanus'. Hobbes of course rejected with notorious virulence the whole tradition of Platonist metaphysics that found in the intelligible the grounds for the sensual. His celebration of the causal character of mathematics depended on redefining demonstration as "a syllogism, or series of syllogisms, derived from the definitions of names to the final conclusion." According to Hobbes, geometry allowed for the clear imposition of

56. *Commentarium de certitudine mathematicarum disciplinarum* ... (Venice, 1569) fol. 95. I have here followed Jardine's translation, "Epistemology" 694. The *Commentarium* was appended to his *In Mechanicas questiones Aristotelis.*

57. This was the line taken for example by Francesco Barozzi in his *Quaestio de certitudine mathematicarum,* appended to his Latin translation of Proclus' *Commentary* (Padua, 1560); cf. Jardine, "Epistemology" 695; G.C. Giacobbe, "Francesco Barozzi e la *Quaestio de certitudine mathematicarum,*" *Physis* 14: 357–74.

58. This was the route taken by the Jesuit Pietro Catena, *Oratio pro idea methodi* (Padua, 1563) 4; cf. Jardine, "Epistemology" 696. See also Benito Pereira, *De Communibus omnium rerum naturalium principiis et affectionibus libri XV* (Rome, 1576) 24; this portion of Pereira's text is translated in Mancosu, *Philosophy of Mathematics* 13. See also Alistair Crombie, "Mathematics and Platonism" 65–68; William Wallace, *Galileo and his Sources: the Heritage of the Collegio Romano in Galileo's Science* (Princeton, 1984).

59. *Exercitationes Paradoxicae adversus Aristotelicus,* ed. Bernard Rochot (Paris, 1959) III: 209, cited in Mancosu, *Philosophy of Mathematics* 23. Mancosu adduces evidence that Barrow had Gassendi's position explicitly in mind.

60. The dispute was with the famous mathematician John Wallis, whose *Mathesis Universalis* (1657) Hobbes attacked, not least for its espousal of algebra as the *mathesis universalis*; *Examinatio et Emendatio Mathematicae hodiernae* (London, 1660); Mancosu, "Aristotelian Logic" 241–65.

names through definition because its objects could be easily perceived by the senses as constructed through motion. But all talk of comprehending such objects at the level of universal intelligibles was to be banished. With Hobbes, we have arrived at a position in the debate furthest from its roots in Proclus.[61]

BARROW AND THE CERTITUDE DEBATE

This divergent lineage of Proclus' position makes clear how pronounced and separate the two sides of his account of mathematical being and knowing could become. The one insisted on the separation from sense, the other stressed a connection to it. And these were debates that Barrow was not only clearly aware of, but in response to which he explicitly developed his own position. If Barrow saw himself as between Plato and Aristotle in his account of the "original of all natural science," he also sought a position between Blancanus' intellectualism and Hobbes' materialistic nominalism.

From that standpoint, we can see what moves Barrow's thinking in his treatment of geometrical abstraction discussed above. That account, and its subsequent articulation throughout the *Lectiones,* sought to keep the intelligible and the sensible aspects together in a single notion of mathematical experience. Hobbes' emphasis on sensible constructive procedures in geometry represented one side. Blancanus' appeal to the place of mathematicals in thinking represented the other. Barrow mediated their positions precisely in his insistence on keeping both sides together in a single notion of experience. In a general sense, Barrow thus returned the debate back to its source in Proclus. But Barrow's return to Proclus is in the form of an assumption: Barrow in effect assumes the Proclan *result* of a mediation by bringing the two sides together while still distinguishing them. At the same time, his flattening of the hierarchy essential to Proclus' system transforms fundamentally what he assumes from Proclus. What Barrow assumes from Proclus, and how he transforms it, depend on one another.

To see how Barrow both joins and keeps distinct the two sides, one need only remember his account of the relation of pure geometry to the mixed sciences. There he flattened a notion of hierarchy of intelligible to sensible, but he did so in order to maintain in the mixed mathematics the precision of mathematical thinking enjoyed in pure geometry. To express this he asks

61. Hobbes, *Emmendatio* 1; *Leviathan, or the Matter, Forme & Power of a Commonwealth Ecclesiastical and Civill* (London, 1651) 1.1, 4. This approach of Hobbes' is neatly surveyed by Douglas Jesseph, *Squaring the Circle: The War between Hobbes and Wallis* (Chicago, 1999) 131–88, and with particular reference to Hobbes' explicit departure from Proclus, 76–85. Barrow had read Hobbes' accounts of these questions, at times criticizing, at times defending him. This is surveyed in Stewart, "Authorized Reason and Reasonable Authority" ch. 6 and 7; see also Mancosu, *Philosophy of Mathematics* ch. 1.

rhetorically: "who has not seen with his eyes or felt with his hand the singular dimensions of bodies?" Appealing to *Posterior Analytics* I.24 and *Metaphysics* II.3, Barrow asserts his position that "mathematics deals with intelligible things and sensible things, since clearly none of its objects is not both intelligible and sensible, in a different respect."[62] In an account of the certainty of mathematics in lecture IV, which reminds one of Piccolomini's position, and which is repeated throughout the *Lectiones*, Barrow will claim that "geometrical quantities are most familiar to us, exposed to [or by] the senses, represented by clear examples, and most easily apparent to the intellect" (*LM* 66).

In that context, Barrow asserts—in a way Proclus would have found repugnant—that what is attributed to the intelligible in geometry is "potissimo jure" attributable to its sensible experience. This being said, it is clear for Barrow that it is not *qua* sensible that geometry owns this feature. The various mixed mathematics are not "parts of mathematics" but rather "examples of geometry," once one has "stripped their propositions from their particular circumstances."[63] Later on in lecture V, in explicit discussion of the question of the certainty of mathematics, Barrow will argue for an even further priority of the intelligible. Here we have Barrow seeking to make both intelligible and sensible aspects inseparable, whilst keeping them distinguished. We see this in his response to the attack of the Skeptics, who questioned the certainty of mathematical objects arising from induction from sense experience. Barrow first sidesteps the attack by appeal to a criterion of interiority. Mathematics enjoys a certainty "of which it suffices that we are intimately conscious ... in the indelible character of our minds ..., certain in the testimony of our own conscience." And although he will mention (but pass over) the "*semina veritatis*" of the "Platonists and the Stoics," and although he will invoke the authority both of Aristotle and Cicero in defending the trustworthiness of sense experience, he will nevertheless grant only a kind of probability to sense experience, one grounded in 'prudence.'[64] And even though our thinking might "be occasioned by sense experience," nevertheless, "that the objects of sense experience exist and of what sort they are is determined

62. *LM* 38: "Cur alia v.g. scientia sphaeram intelligibilem tractet, alia sensibilem? Cum hae reipsa prorsus eaedem sint, et quoad actum mentis subordinatae; nec intelligibili (hoc est universaliter intellectae) sphaerae quicquam tribui possit, quod sensibili (hoc est singulari cuivis) potissimo jure non accommodari possit, et congruit perfectissime."

63. *LM* 44–45. Proclus does not think of being able to 'strip' away the impurities of the sensible to arrive at objects more available for thought (*Commentary* 12.9ff). Barrow's notion of stripping here ("exutas circumstantias") does nevertheless suggest that he thinks the particularity of sense objects must be removed by an act of the mind.

64. *LM* 82: "Attamen ubi quaevis propositio perpetuae experientiae deprehenditur consentanea (praesertim quae ... circa intimam ipsarum constitutionem pertinere) saltem tutissimum erit, et a nobis exiget summa prduentia, promptum ut ei consensum praebeamus."

by thought alone [*sola ratione*]. For who ever saw or perceived by their senses a perfectly straight line?" (*LM* 83).

How can Barrow maintain this seemingly self-contradictory movement back and forth between both aspects—both sides, as it were, of the *certitudo* debate? He does so by assuming and transforming a Proclan position in terms of the general *result* of Proclus' mediatory system. Barrow seems, further, to assume the language of analogy so crucial to Proclus' relation of the sensible and the intelligible realms.

> By an equality of reason [*pari ratione*] we *see*, on the one hand, that we can draw a line between points, and on the other we *know* that a perfectly straight line can be constituted. We know that a magnitude treated of by mathematics can exist; by a similar reason [*simili ratione*] it is attested to by a single observation of sense. (*LM* 84, emphasis mine)

Barrow's use of terms such as "similis ratio" and "par ratio" is evidently only reminiscent of Proclus' far more precise and extensive account of *analogia.* But it is, in meaning at least, clearly a derivative, and that he uses even a derivative of it here at this critical juncture I take to be significant.

The clearest example of Barrow's simultaneous assumption and transformation of a Proclan position is in his account of the relation between *kinds* of mathematical thinking: axioms, postulates, theorems, definitions, and actual constructive problems. In one sense we have in Barrow's account of axioms a clear priority of the intelligible over the sensible, though these are related, again, by a kind of analogy between them. At the same time, the very epistemic hierarchy that Proclus assigned to axioms and theorems over definitions and problems followed directly from Proclus' hierarchy of the intelligible over the sensible. Barrow, however, explicitly flattened that hierarchy of kinds of thinking.

For example, in his account of "axioms," or in reply to the skeptical attack on the stability of sense impressions, Barrow states that the senses can err, but that to judge distances and discern shapes belongs to reason, not to sense. The error belongs to the latter, not to the former. His explanation of this involves the following distinction:

> the human mind ... has the power of intuiting [*intuendi*] universal propositions by its native faculty [*nativa facultate*] in the same manner [*simili modo*] as sense discerns particular ones .... Such universal propositions our minds, without any previous notion and without a reasoning process [*nulloque discursu*], but by its native power, directly contemplates and finds them to be true. And this ... is called by the name *noiesis*, and this faculty is *nous.*[65]

65. *LM* 81; Jesseph cites this passage as a standard rejection of Hobbes' nominalist dismissal of intelligible versus sensible categories (*Squaring the Circle* 211). I see it more as a corrective of Hobbes' one-sided account; Barrow's relation to Hobbes is more complex than can be characterized as a 'rejection.'

But as if to return from stressing too much the 'intelligible' aspect of mathematics, he immediately brings his hearers back to his account of the *certitudo* question. He points out that his position corrects the view of Blancanus, who held that the mathematician deals only with the ideas in the intellect of man and God. Barrow presents his view, by contrast, as doing away with those "idola rerum nusquam existentium," referring critically to Blancanus (*LM* 84). Having defined *noiesis* as involving *nullus discursus* in the case of intuition of axioms, he proceeds to apply his notion of *discursus* equally to axioms, postulates, theorems, definitions, demonstrations and constructive procedures alike. In doing so he appeals (explicitly) to Hobbes' starkly empiricist and constructivist account of mathematics, *at the same time* as maintaining a relation to *noeisis*:

> The truth of certain propositions is deduced not by preconceived notions, not by the observation of sense, but by the declared signification itself of the terms, and from the manifest possibility of their supposition, agreeable to the witness of sense and to the generation of the object, through a certain implicit and quasi virtual discursus, akin to intuitive notions.

Note that he draws a comparison between what he has defined as *noiesis,* an intuition involving no *discursus,* to this procedure of "virtual" or, as he will later call it, "instantaneous discursus." That these kinds of *discursus* are merely different *degrees* of the same fundamental motion of the mind is what justifies his placing all kinds of mathematical inferences—from axioms to constructions—on the same flattened plane. This was a direct and explicit rebuttal of Proclus' hierarchy of mathematical thinking, and Barrow uses his rather fluid account of 'discursus' to justify it. In doing so Barrow effaces the careful distinctions central to Proclus' reading of Euclid's *Elements,* such as between axioms, definitions and demonstrations, or between theorems and problems. Barrow reduces all forms of thinking in mathematics to a kind of *discursus,* varying only with the degree of effort or discursive motion involved. Our knowledge of more fundamental axioms is no different in kind than those that appear after them in the order of our thinking.[66]

In developing his account of a *discursus* that moves from sensible constructive procedures to those requiring only a "nullus discursus" of the mind, Barrow sees himself as having avoided the extremes of both Blancanus and

66. For example, in defence of Appolonius, who had demonstrated an axiom of equality through a constructive procedure of superposition, Barrow had to defend him (and Hobbes!) against Proclus' censure; *LM* 86–88, 126; Proclus' discussion is at *Commentary* 77–81; 179–83 (censure of Applonius); 195.1–15; 243; cf. also Morrow, trans, *Commentary* xxv. Heath, *Euclid* 224–31, provides an overview of this hotly contested foundation of proofs by superposition in Greek geometry and modern interpretations.

Hobbes. It is a standpoint which sees mathematical thinking as all one *discursus*, a bringing together of the activity of mind and sense in one self-contained experience. At the same time, such an experience contains within it the fundamental distinction between intellect and sense—and even the priority of intellect. It is in the immediate context of affirming *this* kind of experience that he articulated his seemingly 'empiricist' account of "the original of all natural science," described above. We can see why Hume himself did not recognize in that account an adequate empiricism, although he saw its merit. The degree to which Hume was critical of Barrow was the degree to which Barrow remained within the influence of a Proclan account of mathematics—assuming its result, whilst attacking its grounds.

EPILOGUE: BARROW IN HUME'S *TREATISE*

Barrow's approach was important to the early modern philosophy of mathematics because he helped make explicit what a century of mathematicians and natural philosophers had been moving towards—an account of mathematical practice that gave to sensible experience the status of a universal. Among the various moves and changed presuppositions within this account, the one most relevant here is the effect of Barrow's attempt to coordinate singular sense experience with intelligible universal inference in the context of mathematics. What this expresses is, according to Peter Dear, nothing less than the crucial justification for granting the status of *philosophia* to singular (often contrived) experiences—'experiments'—when articulated in mathematical form. It allowed for what John Wilkins expressed as an ideal, and what Newton achieved in fact, namely a "Physico-Mathematicall-Experimentall Learning." That phrase was used as if self-evident by John Wilkins to describe the mandate of the fledgling "Royal Society for the Advancement of Experimental Learning." In fact, the phrase was a concatenation of terms whose relation to one another was unclear—if not contentious—as witnessed, for example, in Robert Boyle's resistance to granting mathematics the status of a philosophy adequate to the study of nature. Barrow's *Lectiones* encouraged a conception of mathematical experience (and of mathematically-conceived physical experience) in terms of an "experimental philosophy." As Dear puts it, Barrow's most famous successor, Isaac Newton, "postulated the actual production of particular phenomena [understood universally] so as to allow the formation of a universal science from singulars."[67] It was this representative character of Barrow's positions in the *Lectiones* that

67. Dear, *Discipline and Experience* 242. For a fuller account of the origin of this ubiquity, and of Barrow's representative character, see Dear, ch. 1 and 8.

made them of interest to David Hume in his *Treatise Concerning Human Nature.*[68]

Hume criticized previous philosophical accounts of mathematics for holding that mathematics was a sensible experience grounded in intelligible thought. These accounts attended insufficiently both to such experience and to the attendant thought. According to Hume, Barrow was not sufficiently "experimental" with his own attention to the experience of doing mathematics on the page before him.[69] There is thus a sense in which Hume's account of mathematics is a further development of Barrow's relation of sense and intellect. The "complex ideas" which Hume counts as the "common subjects of our thoughts and reasoning" are associations of "simple ideas" according to principles of association, notably those that give rise to relations, modes, and substances. Those principles are inherent in the simple ideas themselves, namely (first and foundational for the rest), resemblance, then identity, space and time, quantity and number, degree of quality, contrariety, and lastly, cause and effect.[70] The simple ideas, in turn, are entirely dependent both in time and in character upon the sense impressions that gave rise to them as lingering impressions in either the memory or the imagination.[71] There is therefore in our "thoughts and reasoning," *qua* processes of associations of simple ideas into progressions of complex ones, a "sensible" character to those thoughts and reasonings. They are objects of experience. Moreover, with respect to Barrow's handling of definitions of equality—and attendant demonstrations by the method of superposition—it seems that Hume found Barrow's reliance on constructive methods appealing. Citing the view of "many philosophers," Hume says:

68. *A Treatise of Human Nature,* 2nd ed., ed. L.A. Selby-Bigge, text rev. and variant readings by P.H. Nidditch (Oxford, 1978). On the choice of Barrow for this reason by Hume, see Marina Frasca-Spada, *Space and the Self in Hume's Treatise* (Cambridge, 1998) 133–34; N. Kemp Smith, *The Philosophy of David Hume* (London, 1941) 343ff. Other studies of Hume's reading of Barrow are Frasca-Spada, "Some Aspects of Hume's Conception of Space," *Studies in History and Philosophy of Science* 21.3 (1990): 371–441.

69. I use the word "experimental" in the sense that Hume used in seeking to educate his readers in how to attend to their own thinking. This was signalled already in the *Treatise's* title: *A Treatise of Human Nature: Being an Attempt to introduce the experimental Method of Reasoning into Moral Subjects.*

70. Quality is defined more specifically as "the *degrees* in which they possess" the same quality; *Treatise* 15. The way in which these relations are "inherent" is, of course, difficult in Hume. At *Treatise* 13, he distinguishes his account from a common way of seeing relations as 'qualities' of those ideas. Rather, the relations arise "with" the simple ideas, in experience.

71. *Treatise* 1–7. Indeed, Hume's criterion for distinguishing between memory and imagination is entirely the degree to which the "vivacity" of the original impression is retained.

> all definitions [of equality] ... are fruitless, without the perception of such objects; and where we perceive such objects, we no longer stand in need of any definition. To this reasoning I entirely agree; and assert, that the only useful notion of equality, or inequality, is deriv'd from the whole united appearance and comparison of particular objects. (*Treatise* 637)

But Barrow nevertheless fell under Humes' general censure for the distinction of intellect from sense, and the clear priority of the intelligible in Barrow's formulations. Where Barrow went some way toward bringing the intelligible and the sensible together in his formulations, he nevertheless held them quite distinct and bound to a hierarchical logic, as I have shown. One needs only to look to Barrow's assumption that the universal is an object of intellect, and the particular an object of sense, to see how he falls under the *Treatises'* critique of abstract ideas (I.vii). Hume particularly had in mind Barrow's defence of the infinite divisibility in geometric quantities, based on an appeal to its ultimate intelligibility in the mind of God. It is not only that appeal which bothered Hume. It was also that Barrow saw such an appeal as *consistent* with his further confident conclusion that such infinite divisibility was "established by the firmest arguments and the most evident experiences" of geometrical methods grounded in sense experience.[72] Such a concept was known on the one hand only perfectly by God. On the other hand, it could be established by both argument and 'evident experience.' This made sense to Barrow only within his overall account of the coincidence of sense and intellect in mathematics. Although Hume does not make the point himself with respect to Barrow, such an account fell under Hume's general complaint of mathematicians, who

> pretend, that those ideas, which are their objects, are of so refind'd and spiritual nature, that they fall not under the conception of the fancy, but must be comprehended by a pure and intellectual view, of which the superior faculties of the soul are alone capable. (*Treatise* 72)

Hence with respect to Barrow's account of methods of superposition as both evident to sense experience but existing at the demonstrative level in the intellect, Hume pointed to profound tensions.[73] Hume's remarks have the effect of challenging quite generally Barrow's assumed—and fundamentally transformed—Proclan result. At issue was whether Barrow could hold the relation between intelligible and sensible within a unified, self-contained mathematical experience. Hume's critique clarified Barrow's ambiguous re-

72. *LM* 133, 148. Hume's critique of infinite divisibility is discussed in Frasca-Spada, *Space and the Self* 11–55. I have here adapted Dr. Frasca-Spada's treatment considerably in describing Hume's implicit criticism of Barrow's position.

73. See further Frasca-Spada, *Space and the Self* 128–35.

lation to Proclus: Barrow sought to keep both the intelligible and the sensible together in mathematical experience, while he attacked that which for Proclus made the result thinkable.

If it is right to characterize an empiricist approach to mathematics in Hume as the rigorous subjection of both sense experience and thought to the same plane of experience (while banishing as vain any thoughts not reducible to that plane), one can see in what sense Barrow's position is an early empiricism. However, his retention of a realm of thought, coincident with but distinct in kind from sense experience, preserves a distance and hierarchy that Hume could not accept. It is this that Hume would find philosophically inconsistent in the language of mathematicians of his day, whom Barrow represented. In this sense, Barrow relied on Proclus both for what later empiricism would find congenial and uncongenial in his approach. Barrow's suspended position characterized a nascent mathematical natural philosophy. Barrow's ambiguous relation to Proclus constitutes a moment in the emergence of modern philosophy.