The Use of Practical and Theoretical Geometry in Early Modern Astronomy

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There is a distinct contrast in the use of pure geometry and practical geometry in the study of astronomy in the early modern period. Both Kepler and Galileo took their study of the cosmos from a different angle. They both relied heavily on mathematics, but used them in a slightly different way, particularly geometry. Historically, the study of mathematics has been divided into two schools: pure mathematics and applied mathematics. Geometry, a study in the field of mathematics that focuses on the dimensions and forms of shapes and figures, can be divided in a similar way into pure geometry and practical geometry. In Kepler's astronomical studies he applied the former, where Galileo made use of the latter. Kepler differentiates between circles and ellipses in relation to the orbits of the planets and geometrically improves upon both the Copernican and the Tychonic models of the solar system. Galileo uses geometry in more practical terms. He uses complex geometry to improve the operation of the telescope so that he can better observe the heavens. With his improved telescope, Galileo was able to make more accurate measurements of the sky using geometry and geometric tools. This allowed him to develop his own theories about how the cosmos functions. The developmental advances that both of these astronomers made in the fields of astronomy and mathematics strongly influenced the astronomers that followed them. The accomplishments of these astronomers would not have been possible without the individual approaches that they took to the problems they explored. For this reason, the distinction between practical and pure geometry is important to the study of astronomy in the Early Modern Period.

The distinction between pure and applied mathematics comes from their use. Pure mathematics is "the branch of mathematics concerned with the behaviour and properties of numbers, functions, and other abstract entities and structures, studied for their intrinsic interest rather than for their application to solving problems in the real world."[1] Applied mathematics, however, "is concerned with the application of mathematical methods in practical or functional contexts, or within other subjects."[2] This is to say that pure mathematics is abstracted from the physical world, whereas applied mathematics is the sort of math that one would use in everyday life. In terms of physics, applied mathematics takes the laws suggested by pure mathematics in its pursuit of the underlying foundations and looks at and for new phenomena. In this, applied mathematics can revise laws and theories suggested by pure mathematics and make them more precise.[3]

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[1] "pure mathematics, n.". OED Online, accessed November 22, 2013, http://www.oed.com/view/Entry/267871?redirectedFrom=pure+mathematics.

[2] "applied, adj." OED Online, accessed November 22, 2013, http://www.oed.com/view/Entry/9713?redirectedFrom=applied+mathematics.

[3] Ernest W. Brown, "The History of Mathematics," *The Scientific Monthly* 12.5 (1921): 385-413. http://www.jstor.org/stable/6389

Pure geometry is the geometric equivalent to pure mathematics. Since geometry is a subset of mathematics, the same definitions can be applied using slightly different language. In the whole view of mathematics pure mathematics deals abstractly with the world, as does pure geometry, and because of this form geometry, as its does its parent branch of mathematics, takes on a more theoretical or philosophical take on its subject matter. The use of Geometric models "play a crucial rôle in scientific practice, and [...] a fair amount of that practice consists in modelling."[4] As Buldt et al. points out: "Viewed abstractly, the aim of establishing a 'philosophy of X' is quite similar to finding a 'model for Y' in the sciences: One wishes to gain theoretical insight into (some) aspects of a certain phenomenon by representing them in a specific way."[5] Just as pure mathematics is often more philosophical than applied mathematics, so is pure geometry to its counterpart. Pure mathematics is, in fact, very philosophical because "mathematical theorems are a priori truths about acausal, non-spatio-temporal objects."[6] Theorems in geometry function in the same way as they do in pure mathematics, thus taking on a much more philosophical nature than applied geometry. They describe ideal forms of shapes, and because of this do not play a large role in astronomy, but they are still an important part of the pure geometry category that has been established.

Similar to pure geometry's roots in pure mathematics, so practical geometry shares many of its aspects with applied mathematics. Applied mathematics "is a method of reasoning through symbols by which we can discover the consequences of the assumed laws of nature,"[7] which at first seems like the definition of pure mathematics. Although, "the symbolism which we adopt and the rules we lay down with which to reason are immaterial, provided they are convenient for the objects we have in view"[8] seems very similar to what Kepler has done with his pure geometry, it has a major difference. This is, namely, the fact that "we can never deduce the existence of phenomena through mathematical processes which were not implicitly contained in the laws of nature expressed at the outset in symbols."[9] This means that although we use similar techniques for both pure and applied geometry, their priorities are still split. Applied geometry is applied to things in the physical world and our immediate surroundings and observable phenomena. It interacts with the world instead of describing it passively.

Pure geometry is more philosophical than applied geometry because of the way we use it. Pure geometry is not used to describe natural phenomena in the same way that applied geometry is. Pure geometry interacts with the world and gives applied geometry grounds to work on, as well as editing formed theories proposed by pure geometry. Without applied geometry being employed, pure geometry would not have anything to theorize about in terms of physics and astronomy because it would have nothing to abstract from.

Kepler used pure geometry to deduce that the planets orbit the sun in an ellipse rather than a circle or an oval. The full title of his book is *A New Astronomy Guided by Causes; or, A Celestial Physics Drawn from Commentaries on the Motions of the Planet Mars.*[10] While this title

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[4] Bernd Buldt, Benedikt Löwe, and Thomas Müller, "Towards a New Epistemology of Mathematics," *Erkenntnis* 68.3 (2008): 309-29. http://www.jstor.org/stable/40267360.
[5] Ibid., 324.
[6] Ibid., 310.
[7] Brown, "The History of Mathematics", 387.
[8] Ibid., 387.
[9] Ibid.
[10] Ernan McMullin, "Kepler: Moving the Earth," *HOPOS: The Journal of the International Society for the History of Philosophy of Science* 1.1 (2011): 3-22.

seemingly "[announces] an end to the long separation between mathematical and physical astronomy,"[11] there is still a large separation in the text between mathematical and physical astronomy. Although "it was only a posteriori that Kepler turned artists' pictures into ideal models of objective visual images,"[12] he still did it through theoretical, and not practical, geometry.[13] Kepler built "on the observations of Tycho Brahe, [to give] the laws of the planetary motions,"[14] but "Tycho's observational data were not precise enough to point, unaided, to the elliptical, as opposed to numerous other compatible orbits."[15] It was this imprecision that lead Kepler to the use of abstraction and modeling. In *Harmonia Mundi* Kepler explains his laws of planetary motion. This is an abstract conversation about how the universe works, but it is based on empirical data. The laws of planetary motion could not be conceived by Kepler until he first deduced the nature of the orbits in *Nova Astronomia*.

Kepler struggled with deciding the shape of the orbit because "either the circular orbit must be wrong, or the area rule, or both. If it is only the circle that is wrong, then the orbit must be brought within the circle in the mean distances, and the area reduced there, so that the times are shortened." [16] The Radius rule that is being referred to is "the radius vector from sun to planet sweeps out equal areas in equal times."[17] While studying Mars, "Kepler concludes [in] Chapter 58 [of *Astronomia Nova* that] the orbit [of Mars] is a perfect ellipse, or at least differs insensibly from such an ellipse" after trying various other shapes like circles and ovals. But "in the assumption that the errors in the eccentric equations for the circle and auxiliary ellipse are numerically equal but opposite in sign. This leads to the conclusion that the correct orbit lies precisely in the middle." [18] [19] The ellipse works as an orbital path because it is the mean of the other shapes that did fit perfectly. The ellipse resolved the problem of discrepancy between calculations and the observable phenomena. This solved one of the larger discrepancies in science: that being whether

> one [should] privilege the mathematical system on empirical grounds and fit the physics to it as far as one could? Or should one put natural philosophy (in effect, physics) first and adjust the mathematical constructions to it as closely as was practicable? Which should take precedence: the descriptive/empirical or the physical/explanatory?[20]

The ellipse elegantly takes both the descriptive/empirical and the physical/explanatory and combines them into a sort of harmony.

After establishing that the sun is at the center, planets move in ellipses and equal areas sweep out in equal times, Kepler was left with the problem of what drives the system. Kepler solved this with the radius rule which "is interpreted physically to mean that there is a 'solar force' which pushes the planets in their paths about the sun and which falls off in strength with increasing

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[11] Ernan McMullin, "Kepler: Moving the Earth".
[12] Antoni Malet, "Early Conceptualizations of the Telescope as an Optical Instrument," *Early Science and Medicine* 10.2 (2005): 237-262.
[13] Ibid., 261.
[14] Buldt, "Towards a New Epistemology of Mathematics", 396.
[15] McMullin, "Kepler: Moving the Earth", 13.
[16] Curtis Wilson, "Kepler's Derivation of the Elliptical Path," *Isis* 59.1 (1968): 4-25.
[17] Andrew Lenard, "Kepler Orbits More Geometrico,"*The College Mathematics Journal* 25.2 (1994): 90-98.
[18] Wilson, "Kepler's Derivation of the Elliptical Path", 21.
[19] Ibid., 18.
[20] McMullin, "Kepler: Moving the Earth", 4.

distances from the sun."[21] The solar force works by the sun "swinging the immaterial species it emits into a sort of 'whirlpool,'" as he calls it, similar to Newton's gravity.[22] He describes something even more analogous to Newton's gravity when he says:

> the strength of the attraction is proportional to the bulk of the attracting body; its effect on the body attracted is, likewise, proportional to its bulk since, like any other force, it has to overcome the inertia of that body. The earth's attraction keeps the waters of the sea from rising to the moon, and the moon's attraction shows itself in the phenomenon of the tides.[23]

Kepler also states that the "intensity [of the whirlpool] depends on distance as well as on the original intensity of emission, a whirlpool that urges the planets directly onward in their paths."[24] This both explains why different planets travel around the sun at different speeds and why planets travel at different speeds during their individual orbit, solving two major issues with many of the heliocentric models that came before.

These two inferences, "that all planets go round the Sun, with the exception of the moon, which alone has the Earth as its center"[25] and "all the planets are eccentric"[26] are understood as his first two laws, and together they lead to his third law: "the cubes of the major axes are in the same proportion as the squares of the orbital periods,"[27] represented by the mathematical equation $\frac{4\pi^2}{T^2} = \frac{GM}{R^3}$. Kepler relies heavily on his assumption that "the proportions of the apparent motions of individual planets come very close to harmonies"[28] to reach this conclusion. Through his use of abstraction it is easy without further inquiry to conclude that these harmonies are fitted together with the utmost skill so that they support each other mutually as if within a single structure. No single orbit clashes with another, inasmuch as we see that in such a many-sided comparison of their terms harmonies never fail to occur.[29]

Although his abstraction in this is right to an extent, it also allows him to take his assumptions about harmonies and structures too far. One of the most important things to the history of astronomy is the gathering of empirical information of the movement of the planets, and because Kepler has an understanding of "the historical evolution of astronomical theories in an innovative, sophisticated way,"[30] he would have known this. To say that a theory can be easily accepted without further inquiry would go against the tradition that he is a part of.

Kepler takes the empirical data supplied to him from Tycho Brahe and turns it into a model of the way the solar system works. Although his system does work from a mathematical standpoint, there was no way to prove that the planets actually behaved in the way that he described. This is where the notion of abstraction enters his writing. Even in his study of the harmonics of the planets when he

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[21] Wilson, "Kepler's Derivation of the Elliptical Path", 16.
[22] McMullin, "Kepler: Moving the Earth", 15.
[23] Ibid., 16.
[24] Ibid., 15.
[25] Johannes Kepler, *The Harmony of the World,* trans. E. J. Aiton, A. M. Duncan, and Judith Veronica Field (Philadelphia: American Philosophical Society, 1997), 405.
[26] Ibid., 405.
[27] Andrew Lenard, "Kepler Orbits More Geometrico", 90.
[28] Kepler, *The Harmony of the World,* 425.
[29] Ibid., 431.
[30] Malet, "Early Conceptualizations of the Telescope as an Optical Instrument", 243.

> considers the motions of the planets as eccentric, as if viewed from the Sun, it may readily be understood that if an observer were on the Sun, even though it were in motion, to him the earth, although it were at rest (to make a concession already to Brahe), would nevertheless appear to be going around an annual circuit, placed in between the planets, and also in an intermediate period of time.[31]

He abstracts in two ways. Kepler is mathematically abstracted in his ideas about the model for the solar system and conceptually abstracts himself from the world to justify his mathematics. To explain his point further he asks the reader to abstract themselves from their earthly hold and view the solar system from another angle to better grasp his abstract ideas about the way the solar system works.

In his astronomical investigations, Galileo has a stronger focus on applied geometry than Kepler does. Galileo, unlike Kepler, did not grapple as much with abstraction, but rather with, through the use of his telescope, gathering new data and improving extant empirical data. It has been argued that "Galileo's great contribution to the telescope was his improvement of the instrument"[32] and that he "laid foundations for the study of mechanics and proved the rotation of the earth about its axis, besides making a host of astronomical discoveries with his telescope." [33] Galileo had a sophisticated understanding of the invention he refers to as a 'spyglass;' which, with a firm grasp of optics, allowed him to improve upon the instrument and use it in new ways to compile better empirical observations of the heavenly bodies. [34] [35] In this understanding, he makes use of practical geometry. The telescope only works efficiently as an instrument if the curvature of the lens is properly calculated and constructed. Without this use of geometry Galileo would not have been able to use his telescope to observe many different phenomena including our Moon, the fixed stars, and the moons of Jupiter, now called the Galilean Moons.

Galileo embarks on a discussion about his observations of the Earth's Moon. He begins by describing "the darker part, like a sort of cloud, stains the moon's surface and makes it appear covered with spots,"[36] named the Maria, which is now known to be a geographical feature of the moon's surface. [37] After watching the moon rise and fall he notices that it is "full of hollows and protuberances, just like the surface of the earth itself, which is varied everywhere by lofty mountains and deep valleys."[38] This observation is only possible if "the moon's surface reflects the Sun's rays,"[39] because without the light from the sun filling up the valleys in the particular way it does, Galileo would not have been able to deduce that it was earth like. This revelation leads into a discussion of earth shine, where light from the Sun reflects off the Earth and its atmosphere and illuminates the dark parts of the moon, explaining why we can see it during eclipses and during a new Moon.[40] His observations contrasted the "large school of philosophers [who hold] with regard

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[31] Kepler, *The Harmony of the World*, 404.
[32] Albert Van Helden, "The Telescope in the Seventeenth Century," *Isis* 65.1 (Mar., 1974): 38-58.
[33] Buldt, "Towards a New Epistemology of Mathematics", 396.
[34] Galileo Galelei, "The Sidereal Messenger," in *The Essential Galileo*, ed. Maurice A. Finocchiaro (Indianapolis: Hackett Pub., 2008), 45-84.
[35] Ibid., 50-51.
[36] Galilei, "The Sidereal Messenger", 52.
[37] Ibid.
[38] Ibid.
[39] Ibid., 57.
[40] Ibid., 61-63.

to the moon and the other heavenly bodies"[41] and believed it to be perfect, using observation to destroy theory. By pointing out imperfections on the moon and comparing it to the earth, which these schools of philosophy held to be the epitomized imperfection, Galileo proved that not only the moon but the other heavenly bodies were not perfect as they were believed to be.

Galileo's investigation of the moons of Jupiter takes up the majority of *The Sidereal Messenger.*[42] In this discussion we see mathematics used to discover what something is instead of how something works. This is one of the biggest distinctions between theoretical and practical geometry. Galileo used the empirical data that he was collecting to make a model of how Jupiter functioned in relation to its surrounding bodies. In fact when wondering why "the Medicean Stars, in preforming very small revolutions around Jupiter, seem sometimes more than twice as large as at other times" he prefers a physical explanation rather than a geometric one. Galileo asserted that Jupiter

> seems altogether untenable that they approach and recede from the earth at the pints of their revolutions nearest to and furthest from the earth to such an extent as to account for such great changes, for a strict circular motion can by no means produce those phenomena; and an elliptical motion (which in this case would be almost rectilinear) seems to be both unthinkable and by no means in harmony with the observed phenomena.[43]

In this we see a huge break between the way that Galileo approaches astronomy and how Kepler does. Galileo prefers the earthly explanation of the atmosphere distorting the image of the moons to contemplating a phenomenal explanation about how the moons orbit Jupiter, dismissing that their orbit could be anything other than a circle.

This is not the only place where these two great thinkers deviate from each other. Both Kepler and Galileo, in fact, used telescopes and wrote about how they functioned, but just like their astronomical studies differed greatly, so did their approach to the telescope. In Galileo's view,

> 'Mathematical instruments' measured angles, lengths, times, astronomical positions, and so on, according to well-established, unproblematic principles and theories [...] Sixteenth-century mathematical instruments 'were for doing,' while scientific instruments of more modern times are 'for knowing.' Mathematical instruments [...] solved problems. They did not open discussions, but closed them. While the differences between the two kinds of instruments are clear with hindsight, we must yet stress that the opposition between the new optical instruments and the old mathematical instruments is no historian's construction.[44]

Galileo saw the telescope is a scientific instrument rather than a mathematical one. Telescopes are used for tracking the movement of different celestial objects, but they are incapable, at least in the sixteenth century, of measuring angles of declination and how far any particular object has moved over a series of nights. The distinction between mathematical and scientific instruments creates an even greater divide between the way Kepler and Galileo approach astronomy. Kepler was interested in the theory behind the way things worked and used others astronomical data, collected mainly with mathematical instruments, such as distances traveled and exact positions in relation to the earth, to create models. Galileo, however, used scientific instruments to observe new things and make better observations of things that were already well understood.

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[41] Galilei, "The Sidereal Messenger", 52.
[42] Ibid., 67-84.
[43] Galilei, "The Sidereal Messenger", 82.
[44] Malet, "Early Conceptualizations of the Telescope as an Optical Instrument", 238.

Telescopes are also important "from [a] theoretical point of view," because, "it does not matter whether an eye, or a screen, or just empty space gets the light rays coming out of the ocular lens, because the telescope always produces one geometrical image."[45] Reducing the telescope into angles and a geometrical image allowed Kepler to think about the telescope differently. He "let the burden of explanation fall on the angles made by straight lines that carry light rays to the eye. That is to say, [he] assumed that the lenses in the telescope bend the light rays so that they reach the bottom of the eye under new angles, which are responsible for the enlarged appearance of objects."[46] In this, Kepler viewed the telescope differently than Galileo. Kepler was more interested in how the telescope was used rather than as "the notion of a progressive increase in the data mankind was actually able to observe" as Galileo was.[47]

Galileo and Kepler were both great astronomers, but they each approached astronomy from a different point of view. Kepler used geometry to model the solar system, whereas Galileo used scientific instruments to observe new phenomena. The distinction between theoretical and practical geometry played a large role in defining both Kepler and Galileo respectively. Where theoretical geometry, like pure mathematics, deals with the construction of models from observational data and the use of abstraction, practical geometry gathers observational data and focuses on the way things behave without positing explanations of their cause. This distinction within the study of geometry and astronomy in the Early Modern Period is important to the progression of both fields.

[45] Malet, "Early Conceptualizations of the Telescope as an Optical Instrument", 239.
[46] Ibid., 259.
[47] Ibid., 243.

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